PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

Today 10/28:

1. Brief review of entanglement properties: concurrence, negativity, entanglement entropy, entanglement distillation/dilution
2. Continue Week 10---'Quantum entangles'

Brief review of entanglement properties: concurrence, negativity, entanglement entropy, entanglement distillation/dilution

- Concurrence : only for two qubits $\boxed{|\tilde{\psi}\rangle = -i\sigma_y \otimes -i\sigma_y |\psi^*\rangle}$. $C = |\langle\tilde{\psi}|\psi\rangle|$

$\rightsquigarrow |\tilde{\psi}\rangle\langle\tilde{\psi}| = \sigma_y \otimes \sigma_y |\psi^*\rangle\langle\psi^*| \sigma_y \otimes \sigma_y$

e.g. spin $\frac{1}{2}$ $\uparrow \to \downarrow$, $\downarrow \to \uparrow$) $\hat{T} = \underbrace{e^{-iS_y \frac{\pi}{2}}}_{\text{"y-rotation"}} \underbrace{K}_{\text{complex conj.}}$ (QM, Sakurai)

$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

$C = 2|ad - bc|$

mixed state ρ $\quad \tilde{\rho} = \underline{\sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y}$; "$\rho\tilde{\rho}$" $\to$ find eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$

[∃ a formula]

$C = \max\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\}$ Wootters' formula

- negativity (related "Positive Partial Transpose" PPT) $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\Rightarrow |\Phi\rangle\langle\Phi| = \frac{1}{2}$
$$\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \end{array} \xrightarrow[\text{Transpose}]{\text{Partial}} \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

$\to \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, (-\frac{1}{2}) \rightsquigarrow$ entangled

$\Rightarrow$ Negativity $\equiv 2\left|-\frac{1}{2}\right| = 1$

- entanglement entropy for $|\Psi\rangle\langle\Psi|$ (bipartition $A:B$)

$\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$. $S_v(\rho_A) = -\mathrm{Tr}(\underline{\rho_A} \log \underline{\rho_A}) = -\sum_k \sigma_k^2 \log \sigma_k^2$

$|\Psi\rangle = \sum_k \sigma_k |\hat{k}\rangle_A \otimes |\hat{k}\rangle_B$

- distillation $k \equiv [\,|\,] \overset{\psi\ n}{\underset{\psi}{\equiv}} [\,|\,] \equiv \Phi^+$ (half) $\boxed{E_D = \frac{k}{n}}$ | dilution ψ^n (half) $\equiv [\,|\,] \equiv k\,\Phi^+ [\,|\,] \equiv \psi$ (half) $\boxed{E_D \equiv \frac{k}{n}}$

Entanglement of formation

- Bennett et al. ['96] constructed an average quantity for mixed states called entanglement of formation

$$E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_C(|\psi_i\rangle), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

pure state

- Wootters ['98] has provided an analytic formula of E_F for two qubit states (exact form discussed later)

$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$

$E_F(\rho)$ is fcn of $C(\rho)$ vs. $\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|$

- Applying it to our two-spin problem:

$$H = J\vec{\sigma}^1 \cdot \vec{\sigma}^2$$

$$\rho = \frac{1}{Z}\sum_n e^{-\beta E_n}|n\rangle\langle n| =$$

$$= \frac{1-r}{4} I_{4\times4} + r|\Psi^-\rangle\langle\Psi^-|$$

$$r = (e^{3\beta J} - e^{-\beta})/(e^{3\beta J} + 3e^{-\beta})$$

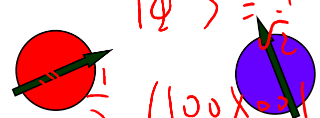
$|\Psi\rangle = \sum_k ... \sum_n ... |\Psi\rangle$

$E_{Cost} = \lim_{n\to\infty} \frac{k}{n}$ very difficult to calculate

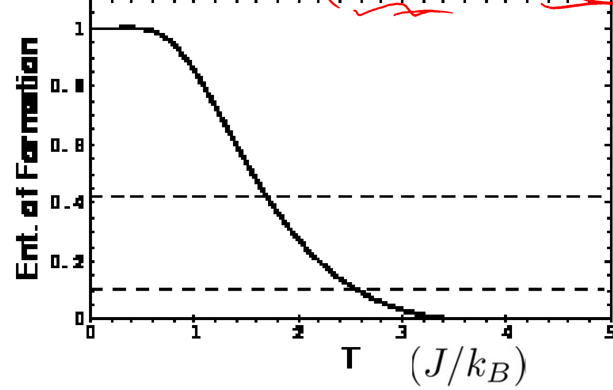
$E_{Dilution}$

= pure state Entanglement entropy

$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$



$\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$



Wootters' formula*

convex hull construction

- Entanglement of formation for mixed states is defined via

$$\boxed{E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_F(|\psi_i\rangle)}, \quad \text{with } \rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$$

$E_c \equiv$ Entanglement cost

$$E_F(|\psi\rangle) = -S_V(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A\,, \quad \text{where } \rho_A = \mathrm{Tr_B}|\psi\rangle\langle\psi|$$

- For two-qubit mixed states, Wootters has found a closed form

heat bath

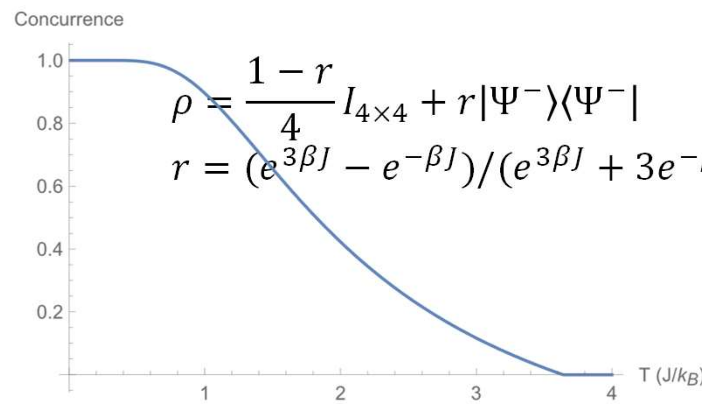
$$E_F(\rho) = H\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right), \quad \text{where } H(x)$$
$$= -x\log x - (1-x)\log(1-x),$$

$H = J\vec{\sigma}_1\cdot\vec{\sigma}_2$

$C(\rho) = \min\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the **concurrence**

where λ_i's, in nonincreasing order, are eigenvalues of

$$\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$



$$\rho = \frac{1-r}{4} I_{4\times4} + r|\Psi^-\rangle\langle\Psi^-|$$
$$r = (e^{3\beta J} - e^{-\beta J})/(e^{3\beta J} + 3e^{-\beta J})$$

Criteria for good entanglement measures* E

[Vedral et al. '97, Vidal '00, Horodecki et al. '00]

1. (a) $E(\rho) \geq 0$; (b) $E(\rho) = 0$, if ρ is not entangled

2. Local unitary transformations should not change the amount of entanglement $\rho \to U_A \otimes U_B \, \rho \, U_A \otimes U_B$

   local (basis transformation)

3. Local operations and classical communication should not increase entanglement $\rho \xrightarrow{LOCC} \{p_k, \rho_k\}$

   - Strong monotone: $\left(\sum_k p_k E(\rho_k)\right) \leq E(\rho)$

     average of entanglements over different outcomes $\leq$ initial entanglement

   - Weak monotone: $E\left(\sum_k p_k \rho_k\right) \leq E(\rho)$

     $\bar{\rho}$ = over different outcomes $\leq$

4. Entanglement cannot increase under discarding information

$$\sum_i p_i E(\rho_i) \geq E\left(\sum_i p_i \rho_i\right)$$

$\bar{\rho}$

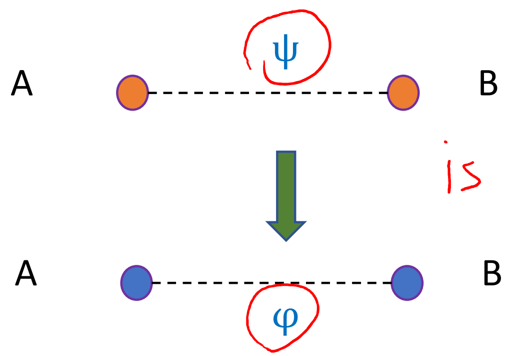
Entanglement transformation for single copy

- We have seen using copies for entanglement distillation and cost. In the limit of infinite copies of bipartite pure states, these two processes are reversible!

$$\lim_{n\to\infty} \quad |\Phi^+\rangle^{\otimes k} \rightleftarrows |\psi\rangle^{\otimes n}$$

reversible

- What if Alice and Bob shares a single copy of ψ ; can this be converted to another state φ (using only local operation and classical communication)?



is this possible ?

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{?} \frac{1}{\sqrt{2}}(\cos\theta|00\rangle + \sin\theta|11\rangle)$$

deterministically

σ_K^2

$(\frac{1}{2}, \frac{1}{2})$ is majorized by $(\cos^2\theta, \sin^2\theta)$

$$\frac{1}{2} \le \cos^2\theta$$

$$\frac{1}{2} + \frac{1}{2} \le \cos^2\theta + \sin^2\theta$$

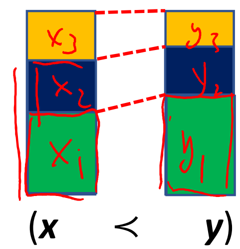
Nielsen's majorization criterion*

[Nielsen, PRL 83, 436 (1999)]

- Majorization: for two sets of numbers ***x*** & ***y*** (e.g. square of Schmidt coefficients in decreasing order), ***x*** is majorized by ***y*** (***x*** ≺ ***y***) if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \text{ for } k = 1, \dots, d$$

$x \prec y$



- Nielsen showed that the transform ψ → φ is possible with probability 1 if and only if the square of their Schmidt coefficients (denoted by λ_ψ, λ_φ) satisfy the majorization relation

$$|\psi\rangle \longrightarrow |\phi\rangle \quad \text{iff } \lambda_\psi \prec \lambda_\phi$$

Note: $p_{\max}(\psi \to \phi) = \boxed{\min_{1\leq m\leq n} \frac{1-\sum_{j=1}^{m-1}\lambda_j(\psi)}{1-\sum_{j=1}^{m-1}\lambda_j(\phi)}}$ *

Example:

$$|\psi\rangle \equiv \sqrt{\frac{1}{2}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$$

$$|\phi\rangle \equiv \sqrt{\frac{3}{5}}|11\rangle + \sqrt{\frac{1}{5}}|22\rangle + \sqrt{\frac{1}{5}}|33\rangle$$

$$|\psi\rangle \not\to |\phi\rangle, \& |\phi\rangle \not\to |\psi\rangle$$

$\left(\frac{1}{2}, \frac{2}{5}, \frac{1}{10}\right)$ $\left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$

$\frac{1}{2} < \frac{3}{5}$

$0.5+0.4 \not< 0.6+0.2$

Another example: Bell state → two-qubit pure state

[Nielsen, PRL 83, 436 (1999)]

- ❑ Bell state $|\Phi^+\rangle = \sqrt{1/2}(|00\rangle + |11\rangle)$

  $|\Psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle,\ (p \geq 1/2)$

$\boxed{|\Phi^+\rangle \rightarrow |\Psi\rangle}$, as $\lambda_{\Phi^+} \prec \lambda_\Psi$

$\frac{1}{2} \leq p$

$\frac{1}{2} + \frac{1}{2} \leq p + 1 - p$

➔ How to achieve this conversion deterministically?

noise channel

$\rho \longrightarrow M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger$

❖ Consider: $M_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ $M_2 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{p} \end{pmatrix}$ $M_1^\dagger M_1 + M_2^\dagger M_2 = I$

$_{anc.}\langle 1| U |\psi\rangle|0\rangle_{anc.} = M_1|\psi\rangle$

$_{anc.}\langle 2| U |\psi\rangle|0\rangle_{anc} = M_2|\psi\rangle$

❑ Alice applies probabilistically either quantum operation M_1 or M_2:

Case 1: $|\Phi^+\rangle = \sqrt{1/2}(|00\rangle + |11\rangle) \xrightarrow{M_1} \sqrt{1/2}(\sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle)$: occurs with prob. = 1/2

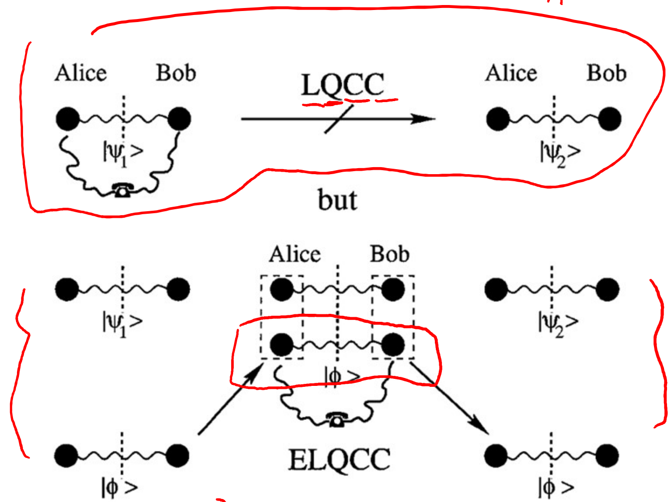
Case 2: $|\Phi^+\rangle = \sqrt{1/2}(|00\rangle + |11\rangle) \xrightarrow{M_2} \sqrt{1/2}(\sqrt{1-p}|00\rangle + \sqrt{p}|11\rangle) \xrightarrow{\text{apply } X_1X_2} \sqrt{1/2}(\sqrt{1-p}|11\rangle + \sqrt{p}|00\rangle)$

Entanglement catalysis

[Jonathan & Plenio, PRL83, 3566 (1999)]

- ❑ Some transformations not allowed (i.e. not with probability 1) by Nielsen's majorization can be achieved if one can borrow some specific "catalytic" entangled state

Nielsen $\lambda_{\psi_1} \not\prec \lambda_{\psi_2}$



Example: $|\psi_1\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle$

$$|\psi_2\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle$$

$$|\psi_1\rangle \not\to |\psi_2\rangle, \text{as } \lambda_{\psi_1} \not\prec \lambda_{\psi_2}$$

However, if they have access to another state:

$$|\phi\rangle = \sqrt{0.6}|44\rangle + \sqrt{0.4}|55\rangle$$

➔ The combined systems have the square of Schmidt coefficients:

$|\psi_1\rangle|\phi\rangle$: 0.24, 0.24, 0.16, 0.16, 0.06, 0.06, 0.04, 0.04

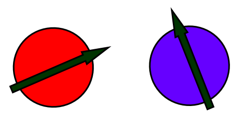
$|\psi_2\rangle|\phi\rangle$: 0.30, 0.20, 0.15, 0.15, 0.10, 0.10, 0.00, 0.00

$|\psi_1\rangle|\phi\rangle \to |\psi_2\rangle|\phi\rangle$ is allowed

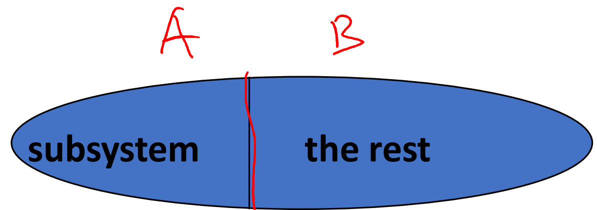
$\lambda_{\psi_1 \otimes \phi} \prec \lambda_{\psi_2 \otimes \phi}$

Bipartite vs. Multipartite

- Concurrence and entropy approaches are essentially bipartite

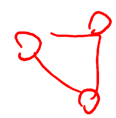


concurrence



entanglement entropy

- Multipartite measures



$|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$

  - Vedral ‘04: relative entropy of entanglement
  - Wei & Goldbart ‘03: geometric measure of entanglement

Multipartite entangled states

Examples: 3-qubit states--- GHZ and W states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Saw this in our discussion of violation of a classical realistic theory using a single shot

$$X \otimes X \otimes X|\mathrm{GHZ}\rangle = (+1)|\mathrm{GHZ}\rangle$$
$$Y \otimes Y \otimes X|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$
$$Y \otimes X \otimes Y|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$
$$X \otimes Y \otimes Y|\mathrm{GHZ}\rangle = (-1)|\mathrm{GHZ}\rangle$$

- For classical local theory, one attributes this to local properties:
  $x_1x_2x_3=+1$, $y_1y_2x_3=-1$, $y_1x_2y_3=-1$, $x_1y_2y_3=-1$
  (where x,y= ±1) ➔ 1 = -1 ! (contradiction)
  But QM: -1 = -1 (consistent)

not even probabilistically

❖ A state that is not equivalent to GHZ state:

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \nrightarrow |GHZ\rangle$$

for a single copy $|W\rangle \nrightarrow |GHZ\rangle$

➔ How do we quantify their entanglement?

Relative entropy of entanglement*

how to parametrize the space unentangled space?

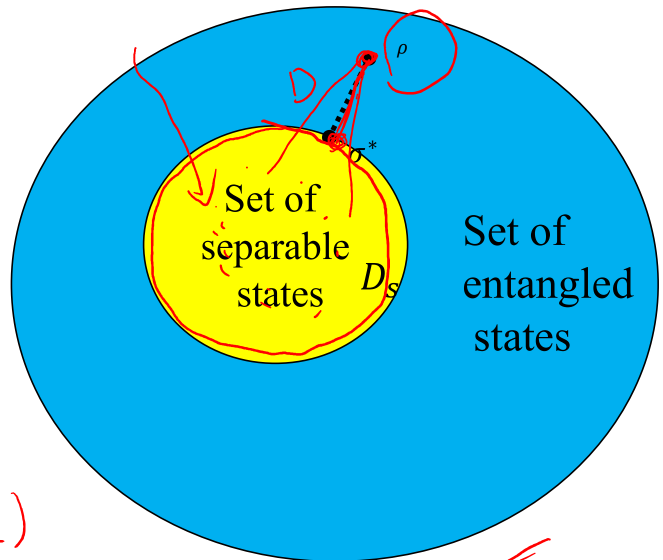
[Vedral et al. '97]

- Define entanglement via relative entropy:

$$E_R(\rho) \equiv \min_{\sigma \in D_S} D(\rho \| \sigma)$$

$$D(\rho || \sigma) \equiv \mathrm{Tr}[\rho \log_2(\rho) - \rho \log_2(\sigma)]$$

vs. $-\mathrm{Tr}\, \rho \log \rho$ $\quad = \mathrm{Tr}(\rho \log_2 \frac{\rho}{\sigma})$



➔ Hard to compute, even for pure states (e.g. W state)!

?

$E_R(GHZ) = 1$

$\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$

$$D_S = \left\{ \sum_i p_i \, |\phi_i^A\rangle\langle\phi_i^A| \otimes |\phi_i^B\rangle\langle\phi_i^B| \otimes \cdots \right\}$$

$\rightarrow$ GHZ.

$\frac{1}{2}|000\rangle\langle 000| + \frac{1}{2}|111\rangle\langle 111|$

Quantifying entanglement

➢ We will be discussing a several other ways to quantify entanglement:

A. Entanglement of distillation [Bennett et al. '96]

B. Entanglement of dilution or Entanglement cost
and Entanglement of formation [Bennett et al. '97]

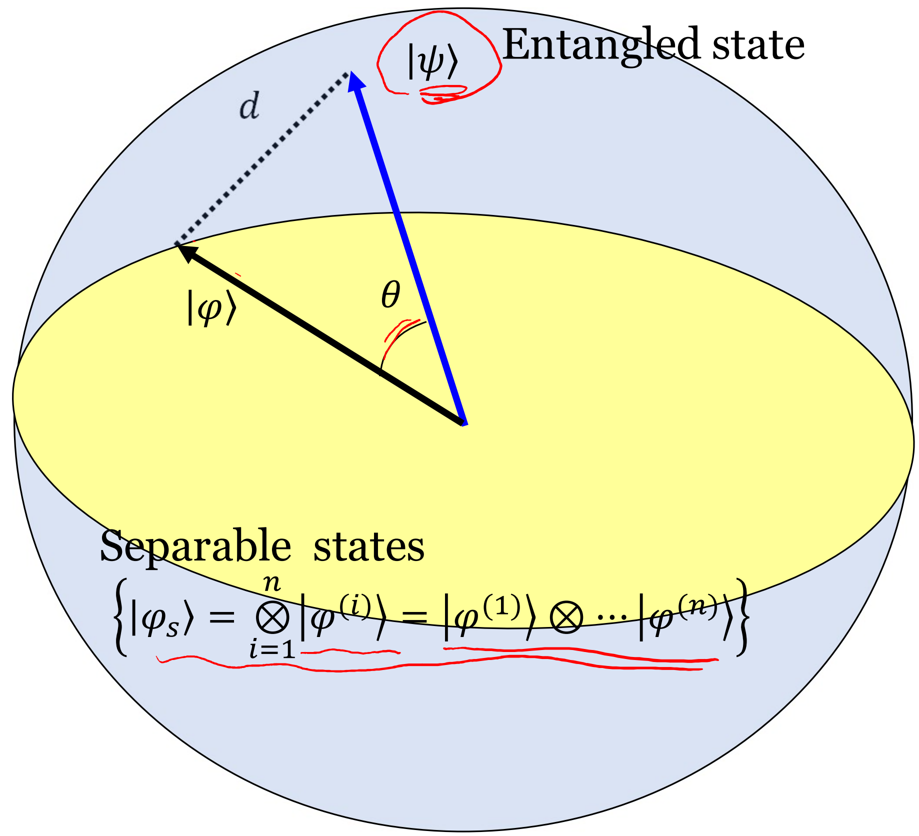
C. A geometric measure for multipartite entanglement
[Shimony '95, Barnum & Linden '01
Wei & Goldbart '03]

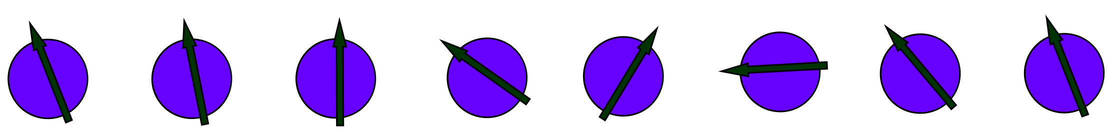
Picture for the geometric measure

Pure states

separable
product state

$$\Lambda(\psi) = \max_{\varphi_s}|\langle\varphi_s|\psi\rangle|_{\max}$$
$$= \boxed{\cos\theta_{\min}}$$





Geometric measure of entanglement

Pure states

[Shimony '95, Barnum & Linden '01]

- A n-partite pure state described by

$$|\psi\rangle = \sum_{p_1 p_2 \cdots p_n} \boxed{\chi_{p_1 p_2 \cdots p_n}} \left|e_{p_1}^{(1)}\right\rangle \otimes \left|e_{p_2}^{(2)}\right\rangle \otimes \cdots \otimes \left|e_{p_n}^{(n)}\right\rangle$$

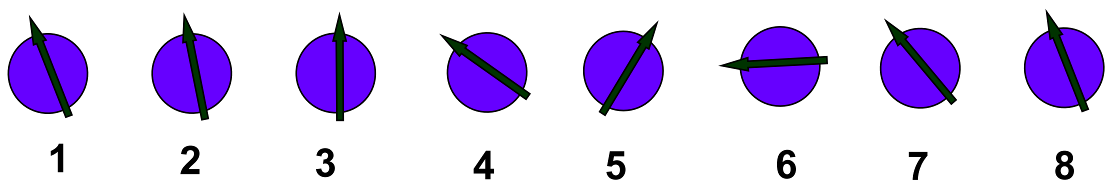
- Find the closest separable (product) pure state

$$|\varphi_s\rangle = \bigotimes_{i=1}^{n} \left|\varphi^{(i)}\right\rangle = \left|\varphi^{(1)}\right\rangle \otimes \cdots \left|\varphi^{(n)}\right\rangle$$

$$\Lambda_{max}(\psi) = \max_{\varphi_s} |\langle \varphi_s | \psi \rangle|$$

- The larger $\Lambda_{max}(\psi)$ is, the less entangled $|\psi\rangle$ is

Entanglement among partitions



- To study entanglement between two groups: {1,2,3,4} and {5,6,7,8}, take separable state to be

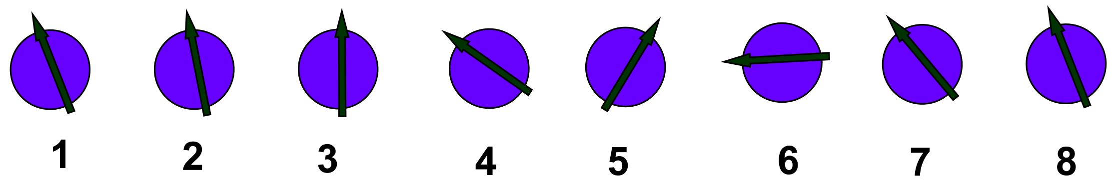
$$|\varphi_s\rangle = \left|\varphi^{(1,2,3,4)}\right\rangle \otimes \left|\varphi^{(5,6,7,8)}\right\rangle$$

and evaluate $\Lambda_{max}(\psi) = \max_{\varphi_s} |\langle \varphi_s | \psi \rangle|$

- To study entanglement among {1,2,3}, {4,5,6}, and {7,8}, take separable state to be

$$|\varphi_s\rangle = \left|\varphi^{(1,2,3)}\right\rangle \otimes \left|\varphi^{(4,5,6)}\right\rangle \otimes \left|\varphi^{(7,8)}\right\rangle$$

Global entanglement



- Making the finest partition in the separable state

$$|\varphi_s\rangle = |\varphi^{(1)}\rangle \otimes |\varphi^{(2)}\rangle \otimes \cdots \otimes |\varphi^{(8)}\rangle$$

we are studying global entanglement of the system

Geometric Measure: Two specific forms

[Wei & Goldbart '03]

1. $E_1(\psi) \equiv 1 - \Lambda^2_{\max}(\psi)$

Bounded by unity; suitable for finite number of parties

2. $E_g(\psi) \equiv -2\log_2 \Lambda(\psi)_{\max}$

this def gives a lower bound on the $E_R(\psi) \geq E_g(\psi)$

No upper limit; suitable for arbitrary number of parties, useful for large N

For certain states $E_R(\psi) = E_g(\psi)$

Entanglement per particle

$$\mathrm{E}(\psi) \equiv \lim_{N\to\infty} \frac{1}{N} E_g(\psi)$$

Examples of bipartite and tripartite pure states

$|\phi\rangle$ = product

$(\sqrt{c}\,|0\rangle + \sqrt{1-c}\,|1\rangle) \otimes (\sqrt{c}\,|0\rangle + \sqrt{1-c}\,|1\rangle)$

Ex.1 $|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$

$$\lambda_{max} = \cancel{\frac{1}{\sqrt{2}}} * \max(\sqrt{p}, \sqrt{1-p})$$

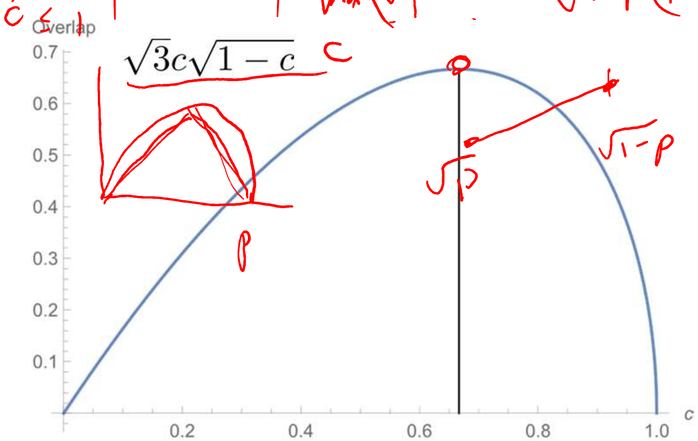
$|00\rangle$ $|11\rangle$

$$C = 2\sqrt{p}\sqrt{1-p}$$

max $|\langle\phi_c|\psi\rangle| = \max(\sqrt{p}\cdot c + \sqrt{1-p}(1-c))$

$0 \le c \le 1$

Note C is the concurrence of Wootters



Ex.2 Tripartite pure states

$\theta = \frac{\pi}{4}$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad \boxed{\lambda_{max} = \frac{1}{\sqrt{2}}} = c\text{-}\theta$$

closest product →

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \quad \lambda_{max} = \frac{2}{3}$$

$\left(\sqrt{\tfrac{2}{3}}|0\rangle + \sqrt{\tfrac{1}{3}}|1\rangle\right)^{\otimes 3}$

$|\phi\rangle = (\sqrt{c}\,|0\rangle + \sqrt{1-c}\,|1\rangle)^{\otimes 3} \longrightarrow (\sqrt{c})^2(\sqrt{1-c})(|001\rangle + |010\rangle + |100\rangle) + \cdots +$

$\langle\phi|\psi_W\rangle = \frac{3}{\sqrt{3}}\, c\, \sqrt{1-c}$ $\quad \max_c \Rightarrow \frac{2}{3}$

GME: examples multi-partite pure states

- N-qubit pure states (e.g. 3-qubit) $|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

  belongs to permutation invariant states:

$$|S(n,k)\rangle \equiv \sqrt{\frac{k!\,(n-k)!}{n!}} \sum_{permutations} |\underbrace{0\cdots0}_{k}\underbrace{1\cdots1}_{n-k}\rangle \qquad \Lambda_{\max}(n,k) = \sqrt{\frac{n!}{k!(n-k)!}}\left(\frac{k}{n}\right)^{k/2}\left(\frac{n-k}{n}\right)^{(n-k)/2}$$

Eg. $\frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$

$\# 2 : 3$

- Interestingly, for these states, we can easily calculate their relative entropy of entanglement

[Wei et al. QIC4, 252 (2004)]

higher ent.

$$E_R = -2\log_2 \Lambda_{\max} \qquad E_R(W) = -2\log_2(2/3) \approx 1.16833, \quad E_R(\mathrm{GHZ}) = 1$$

in other measures it may happen $E(GHZ) > E(W)$

Geometric measure of entanglement and one-way QC

Too much entanglement is useless (Too little entanglement is also useless)

- Gross, Flammia, and Eisert (David Gross et al., 2009) found that random states generically have a high amount of entanglement and if the entanglement of a quantum state is too high, then using it for MBQC cannot offer any speedup for computation and is no better than random coin tossing.

  Too high: $E_g(\psi) \equiv -2\log_2 \Lambda(\psi)_{\max} > n - \delta$

- A similar conclusion that random states drawn uniformly from the state space (or in a more technical term, from the Haar measure) are useless for MBQC was reached by Bremner, Mora, and Winter (Bremner et al., 2009).

- Both results suggest that quantum states that are a universal resource for QC are actually rare and that as commented by Bacon, "entanglement, like most good things in life, must be consumed in moderation" (Bacon, 2009).

Geometric Measure: Mixed states via convex hull*

$$E_{mixed}(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_{pure}(\psi_i)\,, \quad \text{with } \rho = \sum_i p_i \,|\psi_i\rangle\langle\psi_i|$$

- The construction is called convex hull;
  Recall E_F uses the same construction
- Convex-hull construction ensures that any unentangled state has *E*=0
- It complicates the calculation for mixed-state entanglement