Unit 06: Dealing with Errors

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In this unit, we discuss error models, quantum error correction, topological stabilizer codes and topological phases, and error mitigations.

Learning outcomes: You'll be able to understand why quantum information is fragile but quantum correction codes can be used to reduce error rates in logical qubits.

I. INTRODUCTION

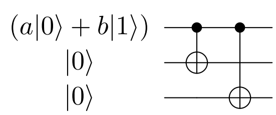
To operate a quantum computer, one needs to actively interact with the quantum device to enact one- and two-qubit gates. One also needs to measure quantum bits to read out the result. It thus impossible to isolate our quantum computer completely from the rest of the world. The interaction with the outside world or the environment can induce errors in our qubits. It was even argued in the beginning of the development that for this reason that quantum computation could not work. For a classical bit, there is only the flip error that flips from 0 to 1 or vice versa with some probability; see Fig. 1a.

But a quantum bit has more errors to occur, e.g. a phase flip: $|0\rangle \to |0\rangle$ but $|1\rangle \to -|1\rangle$. The goal of this unit is to understand how to deal with various types of errors on qubits, so that quantum computation can still persist.

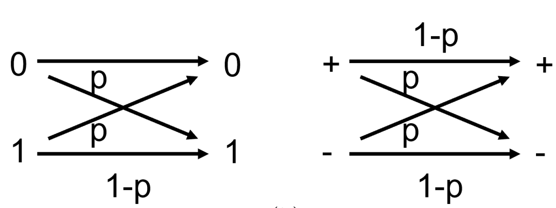
Let us first look at this classical flip channel and how encoding can help to reduce the error rate for the logical bit. We consider the 'redundant' encoding "0"=000, "1"=111; the effect of the bit-flip channel will take 000 to {000: with probability $(1-p)^3$; 001/010/100 with $(1-p)^2p$; 110/101/011: w $(1-p)p^2$; 111: w p^3} and 111 to {111: with probability $(1-p)^3$; 110/101/011 with $(1-p)^2p$; 001/010/100: w. $(1-p)p^2$; 000: w p^3}. We can use the 'majority vote' to recover the original logical bit for some of the cases (e.g. 000 goes to 001/010/100 and the rest 110/101/011 and 111 represents the error. So we have the logical error rate $p' = 3(1-p)p^2 + p^3$. It is obvious from Fig. 2 that $p' < p$ if $p < 1/2$ i.e. the encoding helps. If the original error rate $p > 1/2$, the encoding amplifies the error!

II. QUANTUM ENCODING

Quantum redundant encoding. In the quantum case we have e.g. a quantum bit in an arbitrary pure state $|\psi\rangle = a|0\rangle + b|1\rangle$. One might naively use the redundant encoding as $(a|0\rangle+b|1\rangle)\otimes(a|0\rangle+b|1\rangle)\otimes(a|0\rangle+b|1\rangle)$, as a mapping from $|\psi\rangle|0\rangle|0\rangle$ to this is in general not possible, i.e. no cloning. To overcome this we will use $a|0\rangle+b|1\rangle \to a|000\rangle+b|111\rangle$ and this encoding can be achieved by the circuit



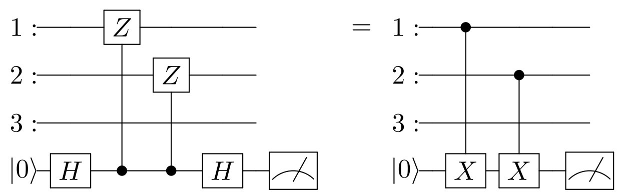
But how do we do the 'majority vote'? We cannot just measure all three qubits in 0/1 as this will collapse the state to a product state, rendering the encoding useful. The solution is to compare them, .e.g by measuring the product of Z_1Z_2 we can compare bits 1&2 to see if they are equal, and simimlary Z_2Z_3 to compare if bit 2 & 3 are equal. By comparing these two results we can figure which qubit has encountered a flip error (of course, this assumes the error rate is small). Then we can apply an X to flip the erroraneous qubit back to the correct state.



(a) (b)

FIG. 1. (a) Bit flip channel. (b) Phase flip channel.

How to measure Z_1Z_2 without measuring separately Z_1 and Z_2? The folloing circuit achieves this,



Here we can introduce the notion of stabilizer operators: Z_1Z_2, Z_2Z_3 and Z_3Z_1 are examples as they 'stablize' the state:

$$Z_1Z_3|\psi\rangle = Z_2Z_3|\psi\rangle = Z_1Z_3|\psi\rangle = |\psi\rangle.$$

Phase flip error. We have more errors than just flipping the bits. Let us consider the phase flip mentioned above: $|0\rangle \to |0\rangle$ but $|1\rangle \to -|1\rangle$, which on any of the qubit will cause the redundant encode to flip from $a|000\rangle + b|111\rangle$ to $a|000\rangle - b|111\rangle$. The above stabilizer measurement cannot detect this error. How can we detect the error?

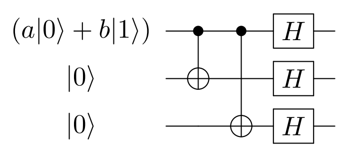
To do that we need to use a different encoding and from the duality between X and Z via the Hadamard gate H, we know that

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle, \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle,$$

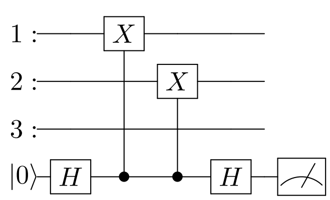
in the $+/-$ basis the phase flip error corresponds to bit flip $+ \leftrightarrow -$. We can thus construct a similar code, dual to the one above,

$$|`0'\rangle \equiv |+++\rangle,\ |'1'\rangle \equiv |---\rangle,$$

which can be achieved by



Then the analysis goes exactly the same as in the bit flip channel, except the change $0/1 \to +/-$ and $Z \to X$ and the encoding reduces the phase flip error rate if the original rate $p < 1/2$. The stabilizer operators include X_1X_2, X_2X_3, and X_3X_1. The stabilizer measurement, e.g., X_1X_2 can be done by



Shor's 9-qubit code. How can we deal with both errors? Shor employed both and combined the above to a 9-qubit code. We can regard that the code he constructed is to expand the $|+\rangle$ in the phase flip code each to a three-qubit flip code: $|+\rangle \to |000\rangle + |111\rangle$, and arrive at the following,

$$|"0"\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|"1"\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

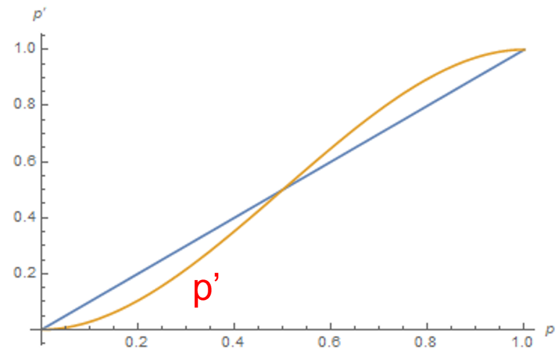
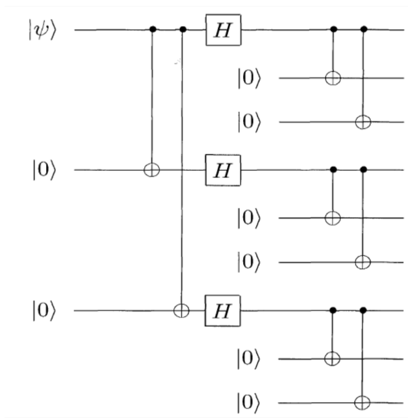


FIG. 2. Illustration of p' as a function of p and the line $p' = p$.

The encoding can be done by the following circuit,



It turns out that this can correct any one-qubit error that is 'any superposition of Pauli errors'.

One can understand this by analyzing the ability to correct bit flip and phase flip of the above two codes that Shor's code inherits. For example, any single bit flip in any of the nine qubits can be detected by running the three-qubit bit flip error detection circuits separately in the three three-qubit blocks: (1,2,3), (4,5,6) and (7,8,9). The phase flip error is slightly different. The previous 3-qubit phase flip code allows determination of which qubit undergoes the phase flip. By generalizing this to the Shor's nine qubits, the flip error can be determined only to within which of the three blocks. The previous stablizer operator X_1X_2 for the three qubits is now generalized to two blocks $(X_1X_2X_3)(X_4X_5X_6)$. The detection circuit now has six CNOT gates controlled from the anicillary qubit instead of two previously.

Given that Shor's code can correct single-qubit X and Z errors, by checking and correcting both, we can use it to correct single-qubit Y errors. We will see below that the ability to correct both X and Z errors guarantees the ability to correct the error due to any arbitrary superposition $\alpha X + \beta Z$.

III. ERRORS ARE QUANTUM OPERATIONS

One can consider errors arise from coupling of the system to the environment that causes joint system to evolve unitarily. For example,

$$U(|\psi\rangle|e_0\rangle) = \sum_k (E_k|\psi\rangle)|e_k\rangle,$$

where the operators E_k acts on the system only and we consider the environment's states $|e_j\rangle$'s being orthonormal, $\langle e_j|e_k\rangle = \delta_{jk}$ and $|e_0\rangle$ is some initial state of the environment.

We note that if one insists, one can consider the environment starts with a mixed state ρ_0, which is the reduced density matrix obtained by tracing out another part of some fictitious environment E', $\rho_0 = \mathrm{Tr}_{E'}(|e_0\rangle\langle e_0|)$. We can

$|e_0\rangle$ is the "purification" of ρ_0. As the unitary U preserve the overlap, for two arbitrary system states $|\phi\rangle$ and $|\psi\rangle$,

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \langle\psi|\phi\rangle = \sum_k \langle\psi|E_k^\dagger E_k|\phi\rangle.$$

Given that it holds for arbitrary $|\phi\rangle$ and $|\psi\rangle$, we deduce that

$$\sum_k E_k^\dagger E_k = I,$$

which is a consequence of probability conservation or trace preserving. (Any lossy operation can also be regarded as ignoring certain outcome in some basis. That is if we include all possibilities, there is no loss.)

We begin the discussion using pure states of the system; what if the system is already in a mixed state ρ? How does it evolve under such unitary U coupling with the environment?

$$\rho \otimes |e_0\rangle\langle e_0\rangle \to U(\rho \otimes |e_0\rangle\langle e_0\rangle)U^\dagger.$$

Given that we do not have access to the environment, we want to know how the system ρ evolve just in terms of degrees of freedom in the system? The answer is

$$\rho \to \mathrm{Tr}_E[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger] = \sum_k E_k \rho E_k^\dagger.$$

The Tr_E indicates a 'partial trace': tracing over only the environment's degrees of freedom. We have seen the partial trace in Unit 01 already.

Let us examine a few error models below.

Bit flip channel.

$$\rho \to (1-p)\rho + pX\rho X^\dagger, \qquad E_1 = \sqrt{1-p}\,I,\ E_2 = \sqrt{p}X.$$

Phase flip channel.

$$\rho \to (1-p)\rho + pZ\rho Z^\dagger, \qquad E_1 = \sqrt{1-p}\,I,\ E_2 = \sqrt{p}Z.$$

Bit-Phase flip channel

$$\rho \to (1-p)\rho + pY\rho Y^\dagger, \qquad E_1 = \sqrt{1-p}\,I,\ E_2 = \sqrt{p}Y.$$

Depolarizing channel.

$$\rho \to (1-p)\rho + p\frac{I}{2}, \qquad E_0 = \sqrt{1-3p/4}\,I,\ E_1 = \sqrt{p/4}X,\ E_2 = \sqrt{p/4}Y,\ E_3 = \sqrt{p/4}Z.$$

For n qubits, the depolarizing channel is

$$\rho \to (1-p)\rho + p\frac{I}{2^n}.$$

Shor's code protects all 1-qubit errors! Now we use the formal description to describe how Shor's code correct all one-qubit errors that are of the form,

$$E_k = e_{k0}I + e_{k1}X + e_{k2}XZ + e_{k3}Z.$$

Let us quote a remark from Nielsen & Chuang:

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The apparent continuum of errors that may occur on a single qubit can all be corrected by
correcting only a discrete subset of those errors.
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The action of E_k on a qubit gives rise to superposition (of no error and three types of errors),

$$E_k|\psi\rangle = e_{k0}|\psi\rangle + e_{k1}X|\psi\rangle + e_{k2}XZ|\psi\rangle + e_{k3}Z|\psi\rangle.$$

We then consider formally using "syndrome" measurements for X and Z errors , and this act of measuring syndromes collapses to either of the four components (as the syndromes are classically distinct) and the corresponding correction operators can be applied to recover $|\psi\rangle$.

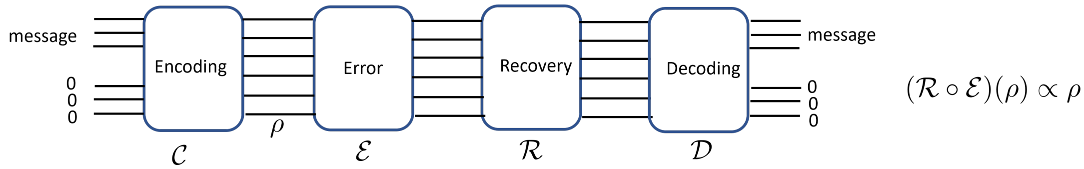

FIG. 3. Illustration of the advantage of error correction codes (i.e. the encoding).

IV. CORRECTABLE ERRORS

Let us consider the process of encoding, error occurrence $\mathcal{E}$, recovery (via syndrome measurement and further operation) $\mathcal{R}$ and the decoding; see Fig. 3. In order for the encoding be useful, the encoded state ρ, after the combination of $\mathcal{E}$ and $\mathcal{R}$, the resultant state must be at least proportional to the original one:

$$(\mathcal{R}\circ\mathcal{E})(\rho)\propto\rho.$$

We will denote by $\mathcal{C}$ the error correction code or the code space and by P the projector to the code subspace. We are interested in what error sets $\{E_k\}$ can be corrected. This can be answered by the following "correctable condition" that the set and the code satisfy,

$$PE_i^\dagger E_j P=\alpha_{ij}P,$$

where α_{ij} is any arbitrary hermitian matrix.

What is the meaning of the above condition? One can simplify this condition by further diagonalizing α: $\alpha_{ij}=\sum_k d_k u_{ik}u^*_{jk}$ and in terms of the new basis,

$$F_k\equiv\sum_j u^*_{jk}E_j,$$

the condition becomes

$$PF_k^\dagger F_l P=d_k\delta_{kl}P.$$

This means that the distinct errors take to the code to orthogonal subspaces and hence they are correctable.

A illuminating exercise is to check this condition in the Shor's code.

- Exercise 10.10 (N&C): Explicitly verify the quantum error-correction conditions for the Shor code, for the error set containing I and the error operators X_j, Y_j, Z_j for j = 1 through 9.

V. STABILIZER GROUP AND ERROR CORRECTION CONDITION

We saw in the 3-qubit bit flip code, there are three stabilizer operators Z_1Z_2, Z_2Z_3 and Z_3Z_1. In addition, the identity I is a trivial stabilizer operator. They form a group $G=\{I,Z_1Z_2,Z_2Z_3,Z_3Z_1\}$, and for any $g\in G$, $g|\psi\rangle=|\psi\rangle$, where $|\psi\rangle=a|000\rangle+b|111\rangle$. This particular group is 'generated' by Z_1Z_2 and Z_2Z_3, i.e. $G=\langle Z_1Z_2,Z_2Z_3\rangle$. Any element in the group G can be obtained by some consequence of multiplying any element in the generator set. Note also that there are two indepedent generators and three qubits; thus we have one logical qubit, and the number $1=3-2$, i.e. the number of qubit minus the number independent generators of the corresponding stabilizer group.

It is useful to define a Pauli group for one qubit:

$$G_1=\{\pm1,\pm i\}\times\{I,X,Y,Z\}.$$

The Pauli group for two qubits is defined as

$$G_2=\{\pm1,\pm i\}\times\{I,X,Y,Z\}\otimes\{I,X,Y,Z\}.$$

The factor $\{\pm1,\pm i\}$ is added such that all the combinations indeed form a group.

Recall the meaning of a group (where is there a group 'multiplication') is defined by the following properties. (1) Closed: for $g\in G$ and $g'\in G$, we have $gg'\in G$. (2) Identity: there exists a special element e_G such that

$ge_G = e_G g = g$. (3) Inverse: for any $g \in G$, there exists an element g' such that $gg' = g'g = e_G$ and the element g' is called the inverse of g. (4) Associativity: for $g_1, g_2, g_3 \in G$, we have $(g_1g_2)g_3 = g_1(g_2g_3)$.

Stabilizer group of Shor's code. It is straightforward to see that the stabilizer group of Shor's code is generated by the following elements: (1) $X_1X_2X_3X_4X_5X_6$, (2) $X_4X_5X_6X_7X_8X_9$ (the two first two from bit flip), (3) Z_1Z_2, (4) Z_2Z_3, (5) Z_4Z_5, (6) Z_5Z_6, (7) Z_7Z_8, and (8) Z_8Z_9 (the latter 6 from the phase flip). Given that there are nine qubits and eight independent generators, we have one logical qubit.

General stabilizer group. Naively, a stabilizer element $g|\psi\rangle = |\psi\rangle$ splits the Hilbert space into two halves and hence reduces the dimension by two. k independent stabilizer elements or generators reduces the dimension from 2^n for n qubits to 2^{n-k}, which corresponds to $n-k$ logical qubits. This is stated in Nielsen and Chuang's book,

- Proposition 10.5 (Nielsen & Chuang): Let $S = \langle g_1 ..., g_{n-k}\rangle$ be generated by $n-k$ independent and commuting elements from G_n (the n-qubit Pauli group), and such that $-I$ is not in S. Then V_S the space stabilized by the group S is a 2^k-dimensional vector space (effectively k qubits).

Note that a stabilizer group is an Abelian group, i.e. any two elements commute $gg' = g'g$. V_S is a k-qubit code space $C(S)$ defined by the stabilizer group S. We can choose two sets of k operators (logical Z's and X's),

$$\{\bar{Z}_1, \dots \bar{Z}_k\},\ \{\bar{X}_1, \dots \bar{X}_k\},$$

such that $\{g_1, g_2, \dots, g_{n-k}, \bar{Z}_1, \dots \bar{Z}_k\}$ are independent and commuting, and

$$[g_j, \bar{X}_l] = 0,\ [\bar{Z}_i, \bar{X}_{j\neq i}] = 0,\ \bar{Z}_i\bar{X}_i = -\bar{X}_i\bar{Z}_i.$$

$\bar{X}$'s are thus logical X operators and $\bar{Z}$'s are logical Z operators. For the case of Shor's code, we can choose

$$\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9, \quad \bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9.$$

Error Correction Conditions for Stabilizer group S. With the understanding of the stabilizer formalism, we can express the error correction conditions in terms of this formalsim. We will not prove it but state the theorem from Nielsen and Chuang:

- Theorem 10.8 (Nielsen & Chuang): (Error-correction conditions for stabilizer codes). Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^\dagger E_k$ is not in $N(S) - S$ for all j and k, where $N(S)$ is the normalizer of S. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.

Note that $N(S)$, normalizer group of S, contains elements E of G_n that preserve S, i.e. for $g \in S$, $EgE^\dagger \in S$. [In this case, $N(S)$ is equal to the 'centralizer' $Z(S)$, the group that commutes with all elements in S.]

Checking Shor's code. For the stabilizer group of Shor's code, $N(S) - S$ [set of elements in $N(S)$ but not in S] contains operators of weight at least three: $X_1X_2X_3$, $X_4X_5X_6$, $X_7X_8X_9$, $Z_1Z_4Z_7$, $Z_2Z_5Z_8$, etc (including $\bar{X}$ and $\bar{Z}$). $E_j^\dagger E_k$ from single-qubit errors are not in this set! Therefore, single-qubit errors can be corrected.

Note that the above set $N(S) - S$ has elements whose weight is at least three, and this number $d = 3$ is called the code distance. So Shor's code uses $n = 9$ qubits to encode $k = 1$ logical qubit with a code distance $d = 3$. Usually, a code distance d can tolerate $\lfloor (d-1)/2 \rfloor$ bits of errors. Shor's code is an $[[n = 9, k = 1, d = 3]]$ quantum error correction code.

VI. OTHER CODES

The stabilizer formalism is very useful. One can define quantum error correction codes using a stabilizer group or its (independent) generators. We give a couple of other codes.

7-qubit Steane code $[[7, 1, 3]]$. The stabilizer generators are

```
IIIXXXX
IXXIIXX
XIXIXIX
IIIZZZZ
IZZIIZZ
ZIZIZIZ
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and we can see that $n = 7$ and $k = 1$, as there are six independent generators. The logical operators (which commute with the generators) are

$$\bar{X} = X_1X_2X_3X_4X_5X_6X_7, \quad \bar{Z} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7.$$

With these one can convince onself that $d = 3$ by considering the minimum weight in the set $N(S) - S$. Similar to Shor's code, the nontrivial operators in the generators are either all X or all Z. These codes are referred to as Calderbank-Shor-Steane (CSS) codes.

One natural question is how to write down the logical basis states in terms of the 7-qubit components? One approach is to use the projector

$$\hat{P} = \prod_{g \in \text{generators}} \frac{I+g}{2} |\psi_0\rangle,$$

and apply $\hat{P}$ to some initial state, e.g. $|0\dots0\rangle$: $\hat{P}|0\dots0\rangle$. This will generate a state that is stabilized by the stabilizer group S. Further logical zero and one are obtained by an additional projection to $(1 \pm \bar{Z})/2$ to this 'ground state'. A general method was proposed by Cleve and Gotteman using efficient quantum circuits: [Phys. Rev. A 56, 76 (1997)].

We thus display the logical states below,

$$\begin{aligned} |0_L\rangle &= \frac{1}{\sqrt{8}}\Big(|000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &\quad +|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\Big), \\ |1_L\rangle &= X_1X_2X_3X_4X_5X_6X_7|0_L\rangle. \end{aligned}$$

Steane code and classical codes. We can have the so-called Parity check matrix H of a classical code C:

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

where the code C is defined by the generator matrix G,

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

such that $HG = 0$.

This is equivalent to $G^T H^T = 0$. So we have the dual code $C^\perp$ defined by the H^T, whose parity check matrix is G^T. One can check (*as an exercise, by examine their columns*) that all code strings generated by H^T are contained in those by G, i.e. $C^\perp \subset C$.

The CSS quantum error correction code is defined by two classical codes C_1 and C_2 with $C_2 \subset C_1$ [in our case: $C_1 = C,\ C_2(= C^\perp) \subset C_1$] via

$$|x + C_2\rangle \equiv \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} |x + y\rangle,$$

where x is any code string in C_1. One can verify directly that the above Steane code can indeed be obtained in this way.

The 5-qubit code $[[5, 1, 3]]$. The stabilizer generators are

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XZZXI
IXZZX
XIXZZ
ZXIXZ
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The logical operators are

$$\bar{X} = X_1X_2X_3X_4X_5, \quad \bar{Z} = Z_1Z_2Z_3Z_4Z_5.$$

The smallest code that corrects one-qubit error!

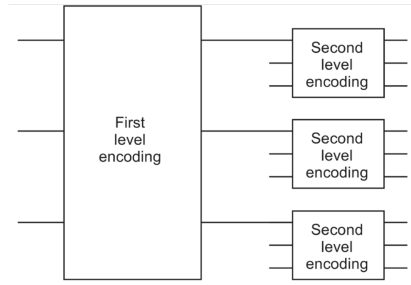

FIG. 4. Illustration of concatenation. [Source? N&C?]

VII. AMPLITUDE AND PHASE DAMPING MODELS

Amplitude damping. The amplitude damping is a model for the spontaneous emission that an atom in the excited state can jump down to the ground state,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \; E_1 \equiv \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

here 0 or up is the ground state and the 1 or down is the excited state; γ is the strength. A qubit described by its density matrix ρ will become

$$\rho \to E_0\, \rho\, E_0^\dagger + E_1\, \rho\, E_1^\dagger =: \mathcal{E}(\rho).$$

For one qubit,

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \to \begin{pmatrix} |\alpha|^2 + \gamma|\beta|^2 & \sqrt{1-\gamma}\,\alpha\beta^* \\ \sqrt{1-\gamma}\,\alpha^*\beta & (1-\gamma)|\beta|^2 \end{pmatrix}$$

We can analyze the efficacy of the error correction model: if it corrects one-qubit (?Pauli?) error, then it should reduce the amplitude damping error rate from γ to γ^2. (But we don't do it here.) In this case, the minimum fidelity is a useful quantity (comparing if the encoded case has better fidelity than the un-encoded case if errors occur), where the fidelity is defined as

$$F = \langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle.$$

Phase damping. The amplitude damping is a model for phase decoherence,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \; E_1 \equiv \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix},$$

where γ is the strength. A qubit described by its density matrix ρ will become

$$\rho \to E_0\, \rho\, E_0^\dagger + E_1\, \rho\, E_1^\dagger =: \mathcal{E}(\rho).$$

For one qubit,

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \to \begin{pmatrix} |\alpha|^2 & \sqrt{1-\gamma}\,\alpha\beta^* \\ \sqrt{1-\gamma}\,\alpha^*\beta & |\beta|^2 \end{pmatrix}.$$

VIII. FAULT TOLERANCE AND ERROR THRESHOLD

If the error probability p of a gate is less than some threshold pth $p_{\rm th}$, then arbitrarily long quantum computations are possible using noisy gates, with a reasonable overhead cost. This uses 'concatenation', i.e. many layers of the same encoding; see Fig. 4. Suppose that each layer encoding reduces the error probability p to cp^2. Then for a total of k layers, the reduction is

$$p \to p^{[k]} \equiv \frac{1}{c}(cp)^{2^k}.$$

We want that the error probability to be smaller than the gate error rate per polynomial number of gates, i.e. $p(k) \leq \epsilon/\text{poly}(n)$. This requires at least that $p^{[k]} < p^{[k-1]}$, and thus

$$p < p_{\text{th}} = 1/c.$$

The exact value of p_{th} and the overhead depend on the error model and the fault-tolerant scheme.

IX. TOPOLOGICAL CODES

The idea of topological codes is to use codes that do not require active error correction but just passive error protection. By now there are many codes, including the Toric code, color codes, Bacon-Shor codes, subsystem codes, fracton codes, Levin-Wen string-net models, etc. They are closely related to intrinsic topological phases of matter, where one key feature is the emergence of exotic excitations called anyons.

Kitaev's toric code. This is one of the earliest topological codes and originally constructed on a torus and later extended to planar surfaces, with the latter called the surface code. However, these codes do not give universal gates and additional work needs to be done in order to make quantum computation on these universal, such as the magic state distillation, which we discuss in a later unit. We will probably postpone the detailed discussions on Kitaev's toric code to the next unit.

X. ERROR MITIGATION (NOT ACTIVE CORRECTION)

Gate error mitigation. This was originally proposed by two groups in two separate works; see Refs. [1, 2]. A gate is achieved by some evolution operator via external field or coupling strength $J(t)$. Ideally, the same area will correspond to the same ideal gate; see Fig. 5. In reality, longer pulses incur more decoherence; we shall use λ to denote the noise strength. For an observable A, we can expand its expectation value in terms of λ,

$$\text{Tr}(A\rho(t=T)) = E_K(\lambda) = E^* + \sum_{k=1}^{n} a_k \lambda^k + R_{n+1}(\lambda, \mathcal{L}, \mathcal{T}).$$

Then we can use different pulses to mimic different noise strengths by varying their lengths: $c_j\lambda$; from several points of c_j, we can extract ideal E^* up to small correction,

$$E^* \approx \sum_{j=0}^{n} \gamma_j E_K(c_j\lambda) + \mathcal{O}(\lambda^{n+1}),$$

where the coefficients γ_j's are chosen such that

$$\sum_j \gamma_j = 1, \ \sum_{j=0}^{n} \gamma_j c_j^k = 0.$$

This method is called Richardson extrapolation.

One much simplified approach that relies on the assumption that single-qubit gates are of high fidelity and the most errors come from the CNOT gate. Then one replace any CNOT gate in the original circuit by $q = (2k-1)$ CNOTs and measure the expectation values of the desired observables. Then one extrapolates to the limit $q \to 0$ to obtain the ideal observable values for the noiseless CNOT limit [3].

Measurement error mitigation. Basic idea: prepare computational states n and measure in computational basis. Gather enough statistics matrix M; see, e.g., Ref. [4]; also a standard procedure in Qiskit.

$$\tilde{P}_{(n_0,n_1,\ldots,n_{N-1})}[\text{measured}] \ = \ \sum_{\vec{m}} \mathbf{M}_{\vec{n};\vec{m}} \, P_{\vec{m}}[\text{ideal}]$$

One only needs to invert the eqution to find $P[\text{ideal}]$ with the constraint that it is non-negative.

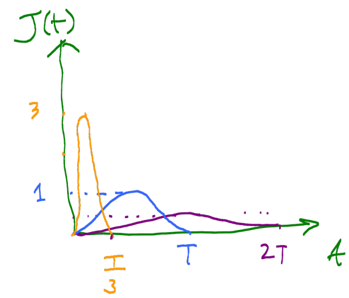

FIG. 5. Illustration of the pulse $J(t)$ and its integrated area.

XI. CONCLUDING REMARKS

In quantum computers, there are more errors than just bit flip and phase. But due to quantum superposition, being able to correct flip and phase errors allows to correct all one-qubit errors, which is an amazing result of quantum error correction. Quantum computers will spend more effort in preventing and actively correcting errors than classical ones; we need measurement to find errors and apply correcting operations. Quantum error correction has been well developed, some of which drawing inspiration from classical coding theory. Now QECC is also used in many other fields, e.g. condensed matter physics and AdS/CFT holographic entanglement.

In this unit, we have discussed error models, quantum error correction, topological stabilizer codes and topological phases, and error mitigations.

It is a good time to check whether you have achieved the following Learning Outcomes:
After this Unit, You'll be able to understand why quantum information is fragile but quantum correction codes can be used to reduce error rates in logical qubits.

Suggested reading: N&C chap 10; KLM chapter 10; Qb chap 5.1-5.2

[1] K. Temme, S. Bravyi, and J. M. Gambetta, Error mitigation for short-depth quantum circuits, Physical review letters **119**, 180509 (2017).

[2] Y. Li and S. C. Benjamin, Efficient variational quantum simulator incorporating active error minimization, Phys. Rev. X **7**, 021050 (2017).

[3] N. Klco, E. F. Dumitrescu, A. J. McCaskey, T. D. Morris, R. C. Pooser, M. Sanz, E. Solano, P. Lougovski, and M. J. Savage, Quantum-classical computation of schwinger model dynamics using quantum computers, Phys. Rev. A **98**, 032331 (2018).

[4] Y. Chen, M. Farahzad, S. Yoo, and T.-C. Wei, Detector tomography on ibm quantum computers and mitigation of an imperfect measurement, Phys. Rev. A **100**, 052315 (2019).