Unit 08: More Topological Please

Tzu-Chieh Wei
C. N. Yang Institute for Theoretical Physics and Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA
(Dated: August 12, 2022)

In this unit, we discuss magic state distillation and urface code quantum computation.
Learning outcomes: You'll be able to understand the most popular topological code (surface code) currently pursued by Google and other groups.

I. INTRODUCTION

In last Unit, we have discussed topological quantum computation by employing anyons. However, it is not easy to find these anyons, and in fact, there is no demonstration of their braiding in experiments despite strong evidence of their existence. We have also seen the simplest yet nontrivial topological model of Kitaev's toric code. The anyons there are abelian and we do not expect the model to provide universal quantum computation. However, other methods exsit, such as the magic state distillation that allows some non-Clifford gates such as the T gate to be injected in an error resistant way.

II. MAGIC STATE DISTILLATION

For this part, we follow the work by Bravyi & Kitaev, PRA71, 022316 (2005) [1].
First, let consider the following 'phase' state,

$$|A_\theta\rangle = (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}.$$

We will see that with this state we can implement the phase gate,

$$U_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

Suppose we have a state $|\psi\rangle = a|0\rangle + b|1\rangle$ and measure the joint operator $Z \otimes Z$ on

$$|\psi\rangle \otimes |A_\theta\rangle = (a|0\rangle + b|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}.$$

This will project the joint system to the +1/-1 eigenspaces of $Z \otimes Z$, which are indicated by the projectors $P^{(+)} = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P^{(-)} = |01\rangle\langle 01| + |10\rangle\langle 10|$, respectively. With probabilty 1/2, the above system is projected to

$$(a|0\rangle + b|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2} \to a|00\rangle + e^{i\theta}b|11\rangle,$$

whose norm square indicates this probability. Adjusting the normalization and applying a CNOT, we have

$$(a|0\rangle + e^{i\theta}b|1\rangle) \otimes |0\rangle = U_\theta|\psi\rangle \otimes |0\rangle,$$

which is the desired action of U_θ on $|\psi\rangle$.
With probability 1/2, instead it is projected to

$$(a|0\rangle + b|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2} \to a|01\rangle + e^{i\theta}b|10\rangle.$$

After applying the CNOT, we have

$$(a|0\rangle + e^{-i\theta}b|1\rangle) \otimes |1\rangle = U_{-\theta}|\psi\rangle \otimes |1\rangle,$$

which is not the desired action, but $U_{-\theta}$ with an opposite sign in the angle.
Magic states $|H\rangle$ and $|T\rangle$. If we have the following state,

$$|H\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle,$$

then we see that it can be made into a 'phase' state with $\theta = -\pi/4$,

$$HS|H\rangle = (e^{i\pi/8}|0\rangle + e^{-i\pi/8}|1\rangle)/\sqrt{2} = e^{i\pi/8}|A_{-\pi/4}\rangle.$$

This allows us to apply a non-Clifford gate $T^\dagger = U_{-\pi/4}$.

In the paper of Bravyi and Kitaev, they also consider the following T state,

$$|T\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle, \ \cos(2\beta) = 1/\sqrt{3}.$$

Can we also use this to do a non-Clifford gate? Suppose we have two copies,

$$|\Psi_0\rangle = |T\rangle \otimes |T\rangle,$$

as in the previous case of the phase state, we measure $Z\otimes$, and only keep the +1 outcome. This occurs with a probability 1/3 and we obtain

$$\cos^2\beta|00\rangle + e^{i\pi/2}\sin^2\beta|11\rangle \sim |\Psi_1\rangle = \cos\left(\frac{\pi}{12}\right)|00\rangle + i\sin\left(\frac{\pi}{12}\right)|11\rangle.$$

Applying CNOT on the two qubits and then Hadamard H on the first qubit, we arrive at

$$(e^{i\pi/12}|0\rangle + e^{-i\pi/12}|1\rangle)/\sqrt{2} = e^{i\pi/12}|A_{-\pi/6}\rangle.$$

This gives us the resource to implement $U_{-\pi/6}$.

We note that $H|H\rangle = |H\rangle$. However, the T state does not have this property, i.e. $T|T\rangle \neq |T\rangle$. Instead,

$$\tilde{T}|T\rangle = e^{i\pi/3}|T\rangle,$$

where

$$\tilde{T} = e^{i\pi/4}SH.$$

We can also define a state that is orthogonal to $|T\rangle$,

$$|T^\perp\rangle \equiv \sigma_y\, H|T\rangle,$$

and can show that

$$\tilde{T}|T^\perp\rangle = e^{-i\pi/3}|T^\perp\rangle.$$

But their density matrices have clear geometrical meaning on the Bloch sphere,

$$|T\rangle\langle T| = \frac{1}{2}\Big[I + \frac{1}{\sqrt{3}}(\sigma^x + \sigma^y + \sigma^z)\Big],$$

and

$$|H\rangle\langle H| = \frac{1}{2}\Big[I + \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z)\Big]$$

Bravyi & Kitaev exploited the 5-qubit error correcting code for T state and CSS codes for H state and constructed an iteratively procedure that takes many noisy copies to distill a single high-quality T and H states, provided, respectively, that

$$\epsilon_T \equiv 1 - \langle T|\rho_{\rm noisy}|T\rangle < \epsilon_0 \approx 0.173, \quad \epsilon_H \equiv 1 - \langle H|\rho_{\rm noisy}|H\rangle < \epsilon_0 \approx 0.141.$$

We will not describe the details here, but refer the readers to their paper. There have also been improvements on their results.

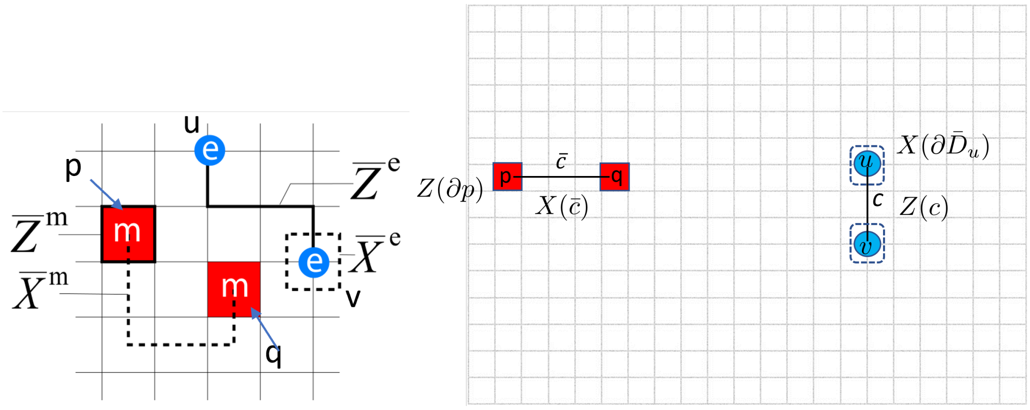


FIG. 1. Illustration of logical operators in the surface code; see e.g. Raussendorf, Harrington & Goyal, NJP 9, 199 (2007).

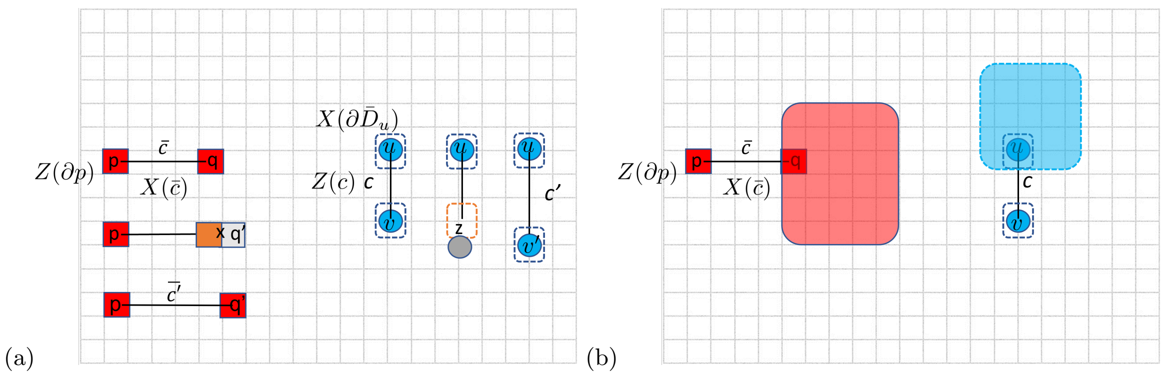


FIG. 2. Illustration of expand and deformation of logical operators in the surface code.

III. SURFACE CODE

For this part of the discussions, we refer the readers to the following references: Bravyi & Kitaev, quant-ph/9811052 [2]; Raussendorf, Harrington & Goyal, NJP 9, 199 (2007) [3] and also Ref. [4]; Folwer et al., Phys. Rev. A 86, 032324 (2012) [5]; Fujii, arXiv:1504.01444 [6].

It is essentially the toric code on a planar geometry and the stabilizer generators are of the form of star and plaquette operators,

$$A_s = \prod_{j\in s} \sigma_x^{[j]},\ \ B_p = \prod_{j\in\partial(p)} \sigma_z^{[j]}.$$

Because there is no nontrivial circles, a logical qubit is encoded in a pair of defects (or holes).

There are two types of logical operators:

- A pair of plaquette defects $\langle p, q\rangle$: B_p and B_q with no constraint separately enforced, but the product B_pB_q is in the stabilizer.

- A pair of star (vertex) defects $\langle u, v\rangle$: A_u and A_v no constraint separately, but A_uA_v is in the stabilizer.

Let us discuss these in more details. For a pair of plaquette defects (a.k.a. *primal defects*), the logical Z is $Z(\partial p) \equiv Z(\partial q)$, where $Z(\partial p)$ is a product of Pauli Zs on the boundary of plaquette p, and the logical X is $X(\bar{c})$, where $X(\bar{c})$ is a product of Pauli Xs on the edges cut by the line $\bar{c}$. One can verify that these logical operators commute with the stabilizer generators and anti-commute between each other.

For a pair of star defects (a.k.a. *dual defects*), the logical Z operator is $Z(c)$, composed of product of Z operators along the curve c. The logical X operators is $X(\partial \bar{D}_u) \equiv X(\partial \bar{D}_v)$, composed of e.g. product of X operators along the edges cut by the boundary of dual plaquette $\bar{D}_u$.

We note that in order for the code to have large code distance, the defects need to be large.

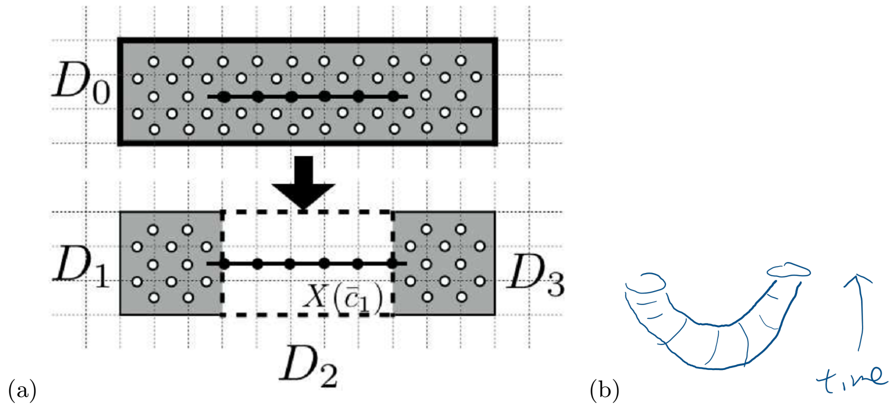


FIG. 3. Illustration of logical X state preparation in the surface code.

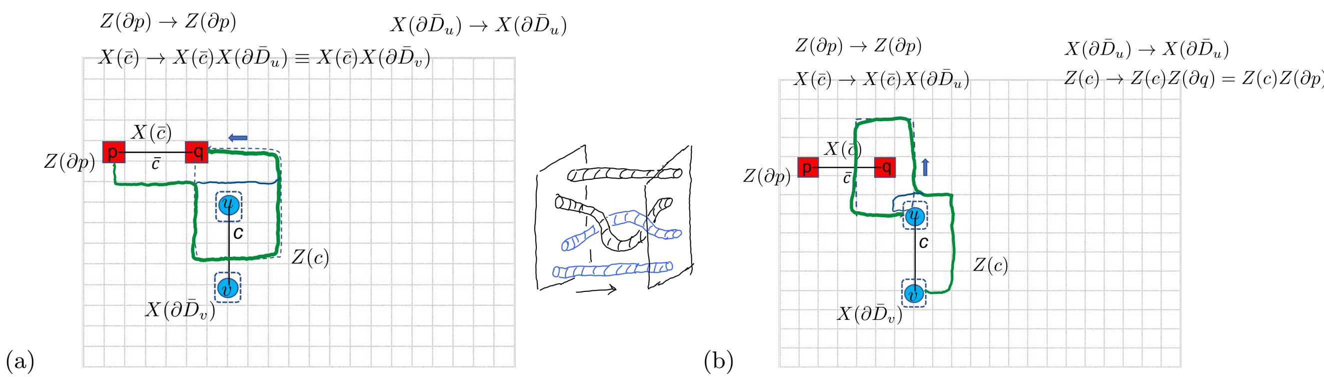


FIG. 4. Illustration of the CNOT by braiding in the surface code.

The logical operators are deformable. For the plaquette type, in the toric code, we apply a string of X operators to move the magnetic excitations. Here, we measure X to expand $Z(\partial q)$. For the star type, in the toric code, we apply a string of Z operators. Here, we measure Z to expand $X(\partial \bar{D}_u)$; see Fig. 2.

Logical Z state preparation.

To prepare a state in the logical X basis, we first create a defect region D_0 by measuring all qubits (circles) in X basis. Then we annihilate defect region D_2 by measuring star operators in D_2; see Fig. 3.

How do we obtain a two-qubit gate, such as the CNOT? It is obvious that we need to perform some kind of braiding. Let consider taking the primal defect q and braiding it around the dual defect u (which is equivalent to braiding u around p). Using the first picture, we see that for the primal logical operators, the braiding transforms them in the following way,

$$Z(\partial p) \to Z(\partial p),\ X(\partial \bar{D}_u) \to X(\partial \bar{D}_u).$$

It is easier to take the latter picture and deduce that for the dual logical operators, the braiding transforms them in the following way,

$$X(\partial \bar{D}_u) \to X(\partial \bar{D}_u),\ Z(c) \to Z(c)Z(\partial q) = Z(c)Z(\partial p).$$

This is the CNOT gate between the primal and dual qubits, using the former as the control and the latter as the target. You will verify this in the homework.

From the above, we obtain a CNOT gate, but unfortunately it has a fixed directionality: from primal to dual qubits. How can we obtain a CNOT gate between two primal qubits or two dual qubits. It turns out we can use 'teleportation' to achieve this; see Fig. 5.

Logical Z rotation: by adding an ancilla in $|+\rangle$ state, perform CZ gates, rotate the ancilla before the Z measurement.

Logical X rotation: bring two defects next to each other and rotate directly on the single qubit adjacent to both defects.

Even though they are not protected, they can be used to inject noisy magic states then distill to a higher-fidelity magic state.

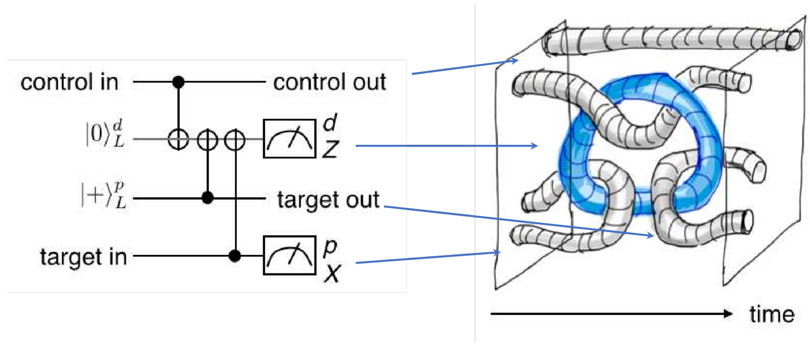
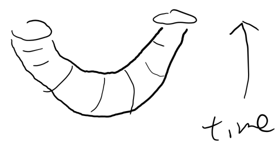


FIG. 5. Illustration of the CNOT between two primal qubits.

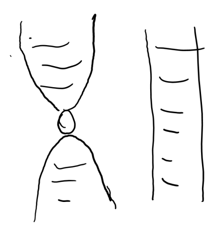
How do we obtain a logical arbitrary phase state? We will illustrate this by diagrams. It is obtained by a rotation around the Z axis on the $|+\rangle$ state,

$$\frac{1}{\sqrt{2}}(e^{-i\theta/2}|0\rangle + e^{i\theta/2}|1\rangle) = e^{-i\theta Z/2}|+\rangle.$$

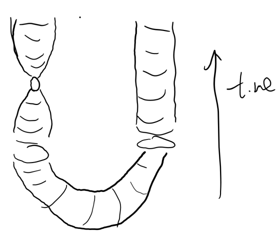
As in the diagram below, we show how to prepare logical $|+\rangle$ state via the X ? measurement mentioned earlier,



Then we do a Z rotation,



Combining both, we have



So far, we have a protected CNOT, but or single-qubit rotations are not protected. As mentioned in the previous section, we can use the magic state distillation. For example, we can use the eigenstate of the Pauli-Y operator and the gate teleportation to implement the S gate; see Fig. 7a. To implement the T gate, we use the eigenstate of the $(X+Y)/\sqrt{2}$ and the gate teleporation to implement the $\pi/8$ gate, a non-Clifford gate necessary for universal quantum computation; see Fig. 7b.

IV. CONCLUDING REMARKS

In this unit, we have discussed the magic state distillation and surface code quantum computation. We have seen that in the surface code, there are naturally protected CNOTs, but other gates need magic state distillation

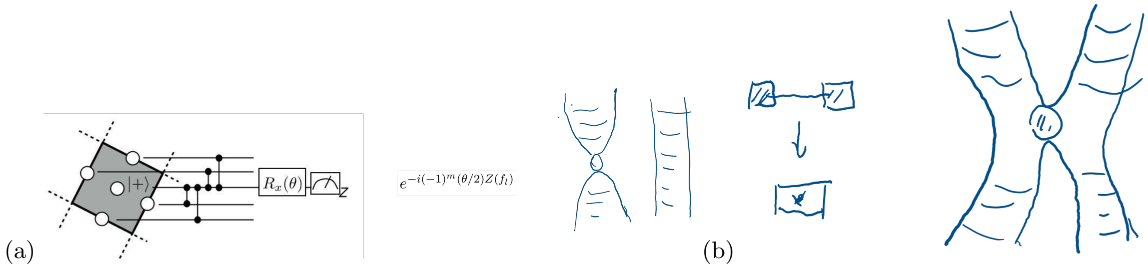


FIG. 6. Illustration of (a) the logical Z rotation and (b) the logical X rotation.

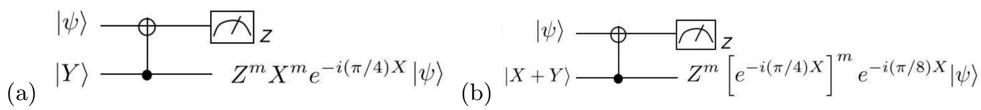


FIG. 7. Illustration of how to implement (a) the S gate and (b) the T gate via gate teleportation.

and can be made of low error rates. It is worth mentioning that many groups are working toward fault tolerant quantum computation, and Google is heavily invested on the surface code and are building devices suitable for its implementation.

It is a good time to check whether you have achieved the following Learning Outcomes:
After this Unit, You'll be able to understand the most popular topological code (surface code) currently pursued by Google and other groups.

Suggested reading: Bravyi & Kitaev, quant-ph/9811052 [2]; Raussendorf, Harrington & Goyal, NJP 9, 199 (2007) [3] and also Ref. [4]; Folwer et al., Phys. Rev. A 86, 032324 (2012) [5]; Fujii, arXiv:1504.01444 [6]; Bravyi & Kitaev, PRA71, 022316 (2005) [1].

[1] S. Bravyi and A. Kitaev, Universal quantum computation with ideal clifford gates and noisy ancillas, Phys. Rev. A **71**, 022316 (2005).
[2] S. B. Bravyi and A. Y. Kitaev, Quantum codes on a lattice with boundary, arXiv preprint quant-ph/9811052 (1998).
[3] R. Raussendorf, J. Harrington, and K. Goyal, Topological fault-tolerance in cluster state quantum computation, New Journal of Physics **9**, 199 (2007).
[4] R. Raussendorf and J. Harrington, Fault-tolerant quantum computation with high threshold in two dimensions, Physical review letters **98**, 190504 (2007).
[5] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland, Surface codes: Towards practical large-scale quantum computation, Phys. Rev. A **86**, 032324 (2012).
[6] K. Fujii, *Quantum Computation with Topological Codes: from qubit to topological fault-tolerance*, Vol. 8 (Springer, 2015).