Unit 13: The Quantum 'Matrix'

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(Dated: August 13, 2022)

In this unit, we discuss quantum simulations and quantum sensing and metrology.
Learning outcomes: You'll be able to get some glimpses to quantum simulations and quantum sensing/metrology and be able to explain them.

I. INTRODUCTION

Richard Feynman at the annual meeting of APS in 1959 said, "Atoms on a small scale behave like nothing on a large scale, for they satisfy the laws of quantum mechanics. So, as we go down and fiddle around with the atoms down there, we are working with different laws, and we can expect to do different things." [1]. More than 20 years later in 1981, he gave a lecture on "Simulating physics with computers" and proposed the idea of quantum simulations [2]. He said, "Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."

He pointed out what we still believe is correct: simulating quantum systems with classical computer requires exponential complexity and proposed to use 'quantum simulators' instead for dynamics:

$$|\Psi(t)\rangle = \text{Time ordered } e^{-i\int_0^t dt' H(t')}|\Psi(0)\rangle.$$

Another topic in this unit is quantum metrology. Heisenberg uncertainty principle [3] is one that is concerning measurement on the position and momentum of a particle, or generally on the two non-commuting observables A and B,

$$\Delta A \Delta B \geq |\langle\psi|[A,B]|\psi\rangle|/2.$$

See the link for the historical documents http://scarc.library.oregonstate.edu/coll/pauling/bond/papers/corr155.1.html.

II. QUANTUM SIMULATIONS

Classical Church-Turing thesis [1936]. Earlier discusssions regarding simulations (of classical systems) were made by Church and Turing, usually referred to as the Church-Turing thesis, which states that, "every function which

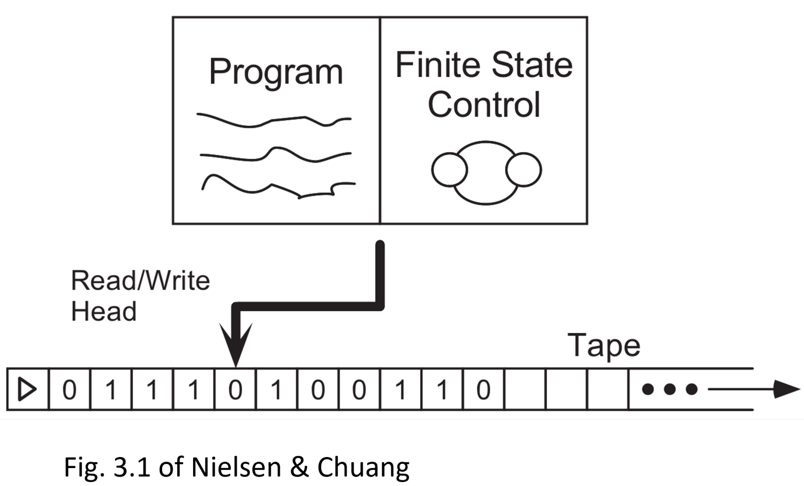


FIG. 1. Illustration of a Turing machine; taken from Fig. 3.6 of the book by Nielsen and Chuang.

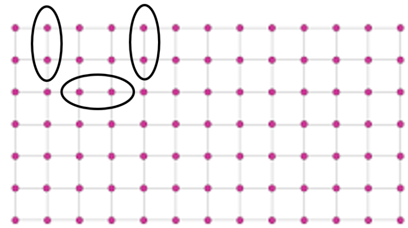


FIG. 2. Illustration of local interactions on a lattice.

would naturally be regarded as computable can be computed by the universal Turing machine." This means that a universal Turing Machine can be used to simulate any other "classical computers". The thesis also defines the notion of an algorithm that computes a function in terms of such a machine. See Fig. 1 for the illustration of a Turing machine.

Later, David Deutsch was interested in how quantum computers could fit into the statement of Church and Turing, and asked "could the laws of physics be used to derive an even stronger version of the Church–Turing thesis?" What he proposed was that a universal quantum computer (or universal quantum Turing machine) is sufficient to efficiently simulate an arbitrary finite, realizable physical system. This is referred to as the Church-Turing-Deutsch Principle or quantum Church-Turing (Deutsch 1985).

Lloyd's quantum simulator. In 1996, Seth Lloyd [4] showed that Feynman's 1982 conjecture is correct that quantum computers can be programmed to simulate any local quantum system (containing few-particle interactions). Lloyd's idea is to evolve a quantum computer in small time steps to allow efficient simulation of time evolution and that the overall time needed grows only polynomially.

In slightly more details, Lloyd considered a set of Hamiltonians

$$\mathcal{S}_H \equiv \{\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_l\},$$

which can be accessible by turning on and off their action on the time evolution of the quantum system. The unitary operators are thus of the form e^{iAt}, where A is in the algebra generated by $\mathcal{S}_H$. The mathematical tool is the Suzuki-Trotter decomposition,

$$e^{iHt} = \left(e^{iH_1 t/n} \ldots e^{iH_l t/n}\right)^n + \sum_{i>j} [H_i, H_j] t^2/(2n) + \sum_{k=3}^{\infty} E(k),$$

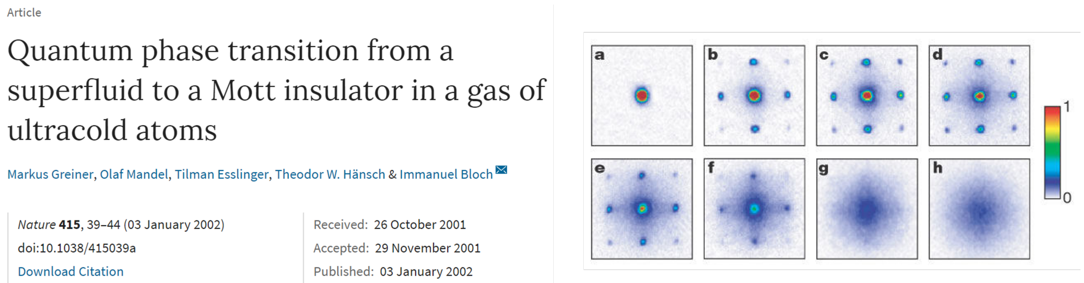
where $E(k)$ contains higher-order corrections. He compared how a quantum simulator works to parking a car, "By going forward and backing up a sufficiently small distance a large enough number of times, it is possible to parallel park in a space only ε longer than the length of the car."

Let us do some counting. For N_q qubits, the state vector for the system's wavefunction has 2^{N_q} components and the evolution matrix is of the size $2^{N_q} \times 2^{N_q}$. Thus, to carry out a simulation of a generic quantum system under time evolution by a classical computer, one needs to spend an exponential time complexity to achieve that.

On the other, using one well-controlled quantum system to simulate another or a quantum model, the complexity is only polynomial in time. Let us assume that each H_j in the accessible set acts on at most dimension m, e.g. on two qubits $m = 4$ (see Fig. 2). The number of operations to simulate $e^{iH_j t/n}$ is m^2 at most. Each term is simulated n times and there are l such terms, so the total number of operation is lnm^2. For a desired total accumulated error ε, the error in each operation should be less than $\varepsilon/(lnm^2)$. If the system has the typical nearest neighbor or next-nearest neighbor interaction, then $l \sim N_q$. From the expansion (in the Trotter decomposition), the first correction term can give an error $\sim t^2/n$ and we set this to be ε and obtain $n \sim t^2/\varepsilon$ for the simulated time duration t. The duration of each opeartion is $t/n \sim 1/t$. So the total discrete real time spent is $l(t/n) * n \sim t$. A quantum simulator's time spent is proportional to the real time duration of the simulated quantum system.

Note in the above we have assumed that the quantum simulator has access to e^{iAt}. If this is not accessible, such as in a typical universal quantum computation, where only certain universal gates are available, we need to further decompose e^{iAt} according to the available gate set.

Cirac-Zoller [2012] criteria for quantum simulations. Similar to the DiVincenzo criteria for universal quantum computation, Cirac and Zoller provided a set of desirata for universal quantum simulations [5].


Article

Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms

Markus Greiner, Olaf Mandel, Tilman Esslinger, Theodor W. Hänsch & Immanuel Bloch

Nature **415**, 39–44 (03 January 2002)
doi:10.1038/415039a
Download Citation

Received: 26 October 2001
Accepted: 29 November 2001
Published: 03 January 2002

FIG. 3. Illustration of an experiment by Greiner et al. [6].

- Quantum system (bosons or fermions with or without spins): it must contain a large number of degrees of freedom. Particles may need to be confined in some region of space. [cf. well-defined two-level systems in DiVincenzo's.]

- Initialization: we need to be able to prepare (at least approximately) a known quantum state for the system. [cf. initialized to e.g. zero in DiVincenzo's.]

- Hamiltonian engineering: It needs to be possible to engineer a set of interactions with external fields or between different particles, with adjustable values. [cf. universal one- and two-qubit gates in DiVincenzo's.] They may involve a reservoir to simulate open-system dynamics. Among the accessible Hamiltonians there should be some that cannot be efficiently simulated (at present) with classical techniques. [This last one is to ensure certain quantum advantage.]

- Detection: We need to be able to perform measurements on the systems. This can be individual (i.e. addressing a few particular sites on the lattice) or collective. Ideally, one should be able to perform single-shot experiments that can be repeated several times; one would be able to determine not only the average of an observable $\langle S\rangle$ but also the average of a function of the observable $\langle f(S)\rangle$. [This corresponds to the readout in DiVincenzo's.]

- Verification: Even though there is in no way of efficiently verifying the result if the simulation size is large, one needs to have a way of increasing the confidence in the result. [We do not have a corresponding criterion in DiVincenzo's list.]

How to achieve the last item, verification, is an important question. So far we have very limited tools. We could for example using exactly solvable models in physics to provide a benchmark. We may also use the simulations in forwards and backwards in time to see if the end result is the initial state. Or we could develop more powerful and efficient numerical simulations, such as the Quantum Monte Carlo or tensor network methods.

III. PHYSICAL SYSTEMS AND RECENT EXAMPLES

Superfulid to Mott insulator transition. Arguably the most well-known post-BEC era pioneer paper is the one by M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a goas of ultracold atoms, Nature **415**, 39 (2002) [6]; see Fig. 3. The authors trap cold atoms (Rubidium 87) in an optical lattice and vary the trapping depth, which effective changes the ratio of the tunneling (of a particle from one lattice site to a neighboring one) and the onsite interaction strength t/U. The system is effectively described by the Bose-Hubbard model,

$$H = -t\sum_{\langle i,j\rangle}\hat{b}_i^\dagger\hat{b}_j + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i-1) - \mu\sum_i \hat{n}_i,$$

which was proposed and studied theoretically way before the experiment.

Topological Haldane model. In cold atoms, not only bosons can be cooled, fermions can also be used to study low-temperature phases. An experimental paper by G. Jotzu, M. Messer, R.Desbuquois, M. Lebrat, T.Uehlinger, D. Greif, and T. Esslinger, Experimental realization of the topological Haldane model with ultracold fermions, Nature **515**, 237 (2014) [7], cools ^{40}K Potassium atoms trapped in a optical lattice. They realize the so-called toplogical

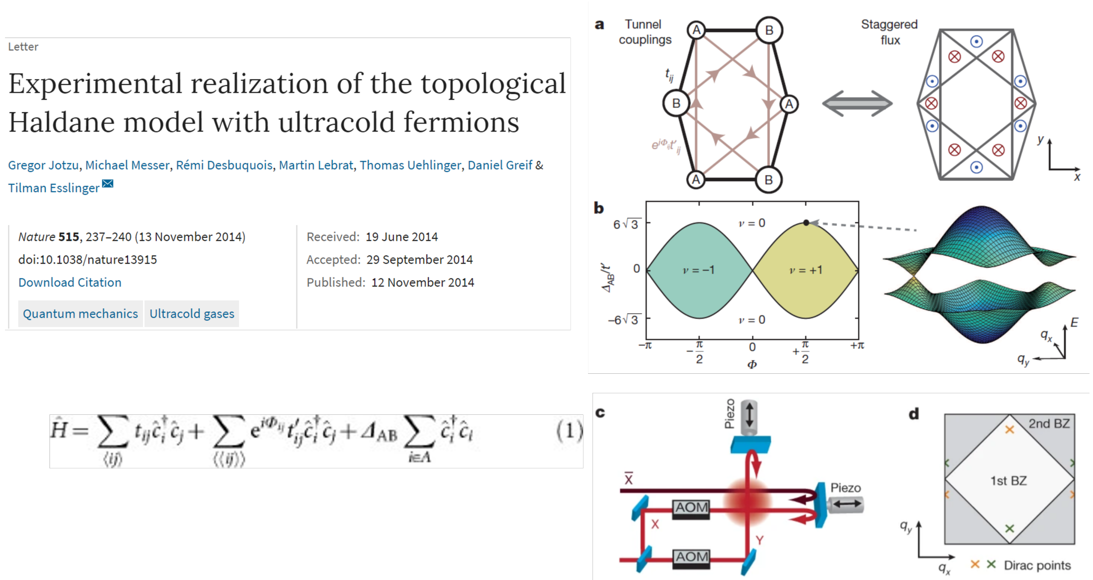


FIG. 4. Illustration of an experiment by Jotzu et al. on the topological Haldane model.

Haldane model, which is a milestone theoreitcal work on the Chern insulator and later topological insulators; see Fig. 4. The model is described by the Hamiltonian,

$$H = \sum_{\langle ij \rangle} t_{ij}\hat{c}_i^\dagger \hat{c}_j + \sum_{\langle\langle ij \rangle\rangle} e^{i\Phi_{ij}} t'_{ij}\hat{c}_i^\dagger \hat{c}_j + \Delta_{AB}\sum_{i\in A}\hat{c}_i^\dagger \hat{c}_i.$$

The above two experiments use **ultracold quantum gases**. They have high degree of controllability, novel detection possibilities and extreme physical parameter regimes (compared to solid-state systems); see Bloch, Dalibard & Nascimbène, Nat. Phys. 2012 [8]. There are other physical systems, such as trapped ions [see Blatt and Roos, Nat. Phys. 2012 [9]], NV diamond centers [10, 11], photonics [12, 13], and superconducting qubits. [14–16].

We list the remaining works mentioned in the lecture:

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- "Engineered two-dimensional Ising interactions in a trapped-ion quantum simulator with hundreds of spins," J. W. Britton, B. C. Sawyer, A. C. Keith, C.-C. J. Wang, J. K. Freericks, H. Uys, M. J. Biercuk, and J. J. Bollinger, Nature **484**, 489-492 (2012) [19].
- "Observation of a many-body dynamical phase transition with a 53-qubit quantum simulator," J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z.-X. Gong, and C. Monroe, Nature **551**, 601–604 (2017) [20].
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- "Spectroscopic signatures of localization with interacting photons in superconducting qubits," P. Poushan, C. Neill, J. Tangpanitanon, V. M. Bastidas, A. Megrant, R. Barends, Y. Chen, Z. Chen, B. Chiaro, et al., J. Martinis, Science **358**, 1175-1179 (2017) [23].
- "Quantum supremacy using a programmable superconducting processor," F. Arute, K. Arya, [. . .], J. M. Martinis, Nature **574**, 505–510 (2019) [24].

IV. QUANTUM METROLOGY

For two non-commuting observables (Hermitian matrices) A and B, they standard devitions obey

$$\Delta A \Delta B \geq |\langle\psi|[A,B]|\psi\rangle|/2,$$

where $\Delta A \equiv \sqrt{\langle A^2\rangle - \langle A\rangle^2}$. The most studied example is the uncertainty between the position x and the momentum p of a particle, $[\hat{x},\hat{p}] = i\hbar$, which implies that $\Delta x \Delta p \geq \hbar/2$. Heisenberg initially regarded this as a relationship between the precesion of a meaurement and the disturbance it creates. However, from the derivation, the correct interpretation should be these are intrinsic uncertainties, namely, "if we prepare a large number of quantum systems in identical states, $\psi\rangle$, and then perform measurements of A on some of those, and of B in others, the statistical uncertainties satisfy above inequality."

Although Heisenberg's original idea is intuitive, it is not rigorous. A recent experiment was carried out by Rozema et al. [Phys. Rev. Lett. 109, 100404 (2012)] [25] that uses weak measurement and shows that the original precision-disturbance relationship can be violated. In classical mechanics, measurement of position and momentum or speed can be made separately and arbitrarily precise and there is no fundamental limit. In quantum mechanics, however, the act of measurement changes the system (i.e. some kind of quantum backaction), and thus repeated measurement cannot obtain initial properties, for example, observing the location of a particle 'localizes' the particle or collasping the particle's wavefunction. That is one reason why the measurement precision and disturbance is not easily quantified simulatenously in an experiment and a weak measurement is needed. But it is possible to formulate some uncertainty relation between measurement precision and disturbance, but one needs to carefully define measurement noise and disturbance and or introduce new inequality, see e.g. Ozwa, Ann. of Physics. 2004 [26] and Hofmann, Phys. Rev. A 2013 [27].

Deriving the uncertainty principle. Consider two Hermitian operators A and B: $A^\dagger = A$ and $B^\dagger = B$. We rewrite

$$AB = \frac{1}{2}(AB+BA) + \frac{1}{2}(AB-BA) = \frac{1}{2}\{A,B\} + \frac{1}{2}[A,B],$$

evaluate the expectation value with respect to a wavefunction $|\psi\rangle$,

$$\langle\psi|AB|\psi\rangle = \frac{1}{2}\underbrace{\langle\psi|\{A,B\}|\psi\rangle}_{\text{real}} + \frac{1}{2}\underbrace{\langle\psi|[A,B]|\psi\rangle}_{\text{pure imaginary}} =: x + y\,i.$$

Thus we have

$$\frac{1}{4}|\langle\psi|\{A,B\}|\psi\rangle|^2 + \frac{1}{4}|\langle\psi|[A,B]|\psi\rangle|^2 = |\langle\psi|AB|\psi\rangle|^2 \leq \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle,$$

where we have used the Schwarz inequality $|\vec{a}\cdot\vec{b}|^2 \leq |\vec{a}|^2|\vec{b}|^2$. By replacing A by $A - \langle A\rangle$ and B by $B - \langle B\rangle$, we arrive at

$$\frac{1}{4}|\langle\psi|[A,B]|\psi\rangle|^2 \leq \langle\psi|(A-\langle A\rangle)^2|\psi\rangle\langle\psi|(B-\langle B\rangle)^2|\psi\rangle,$$

which gives the celebrated uncertainty relation.

Standard quantum limit. Let us consider a particle involving in time. Its wavepacket will expand and the uncertainty in its momentum or veleocity is

$$\Delta v = \Delta p/m \geq \hbar/(2m\Delta x),$$

and the deviation on the position is

$$\Delta x(t) \sim \Delta x + \frac{\hbar t}{2m\Delta x}.$$

If we minimize this with respect to Δx, we obtain the standard quantum limit,

$$(\Delta x)_{\rm SQL} = \frac{\hbar t}{2m(\Delta x)_{\rm SQL}} = \sqrt{\frac{\hbar t}{2m}}.$$

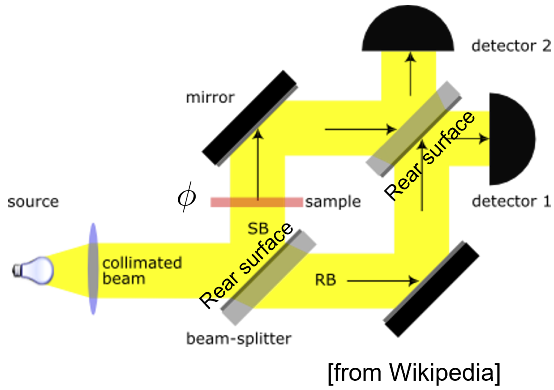

FIG. 5. Illustration of the Mach-Zehnder interferometer and estimation of a phase ϕ; see, e.g., `https://en.wikipedia.org/wiki/Mach-Zehnder_interferometer`.

This seems to suggest a limit on how well one can measure the position.

When we consider the phase and number uncertainty, we have

$$[\hat{\phi}, \hat{N}] = i \ \rightarrow \Delta\phi \geq 1/(2\Delta N) \geq 1/N_{\rm tot},$$

where in the last inequality we have used the $\Delta N \leq N_{\rm tot}$. This seems to suggest the limit on the accuracy to measure the phase, which is usually referred to as the Heisenberg limit: the uncertainty in measuring a phase is limited by the total number of photons used.

A natural question is how can we deal with the uncertainty? [See, e.g. Giovannetti, Lloyd & Maccone, Science 06, 1330 (2004) [28].] First, we can monitor only one out of a set of incompatible observables. But this creates large uncertainty in the other noncommuting observable(s). Secondly, we can employ a quantum state in which the uncertainty in the to-be-monitored observable is small (at the cost of a very large uncertainty in complementary observables even before measurement). But there are also errors in addition to the systematic errors. Typical source of errors are environment-induced noise from vacuum fluctuations (the so-called shot noise) and dynamically induced noise in e.g. the position measurement of a free mass (the so-called standard quantum limit). But these do not seem be a fundamental limit.

Mach-Zehnder interferometer. We have seen the Mach-Zehnder interferometer in Unit 01 in the context of Vaidman's 'interaction-free' measurement. You have also seen that the intensity in the two ports oscillate with the unknown phase, $I_1 = I_0\cos^2(\phi/2)$, and $I_2 = I_0\sin^2(\phi/2)$. This can be verified by measuring the intensity, which is equivalent counting the clicks from photo-detectors. If photons are independent then, the uncertainty in the normalized intensity is

$$\Delta(I_1 - I_2)/I_0 = \sin\phi\Delta\phi \propto 1/\sqrt{N}.$$

The uncertainty in the phase induced by the sample is limited by the number of photons used (i.e. the shot-noise limit). We will come back to this system later using the quantum Fisher information.

One natural question is, "Can the shot-noise limit be overcome?" Such a question was studied long ago, see e.g. Caves, PRD 1981 [29]; Barnett, Fabre, Maitre, Eur. Phys. J. D (2003) [30], where it was shown that by using the squeezed vacuum in the original vacuum (no-input) port, one can an uncertainty

$$\Delta\phi \propto 1/N^{3/4}.$$

Later in the development of quantum sensing and metrology, it was shown that this uncertainty can be further improved using entangled states to the Heisenberg limit, $\Delta\phi \propto 1/N$; see Dowling, arXiv:0904.0163 [31].

The main idea is as follows. If we can create the so-called 'N00N' state, which is a two-mode entangled state,

$$|{\rm N00N}\rangle \equiv (|N_A, 0_B\rangle + |0_A, N_B\rangle)/\sqrt{2}.$$

Then a phase shift in one arm takes the state to

$$(e^{iN\phi}|N_A, 0_B\rangle + |0_A, N_B\rangle)/\sqrt{2}.$$

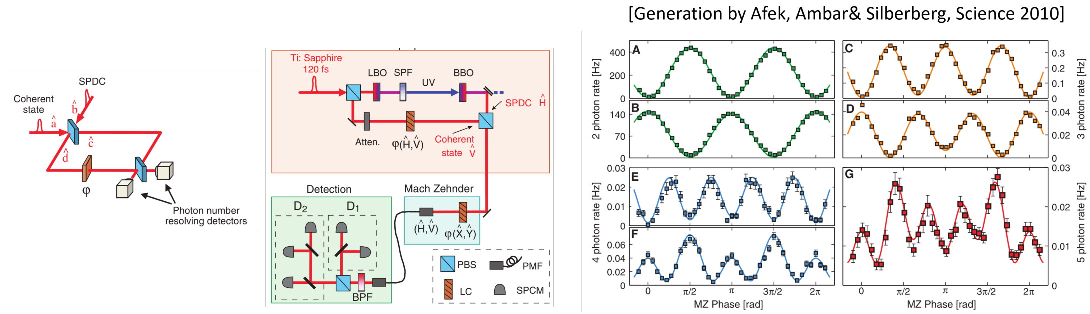

FIG. 6. Illustration of an experiment by Afek, Ambar & Silberberg, Science 2010 [32], to achieve the Heisenberg limit.

If one can interfere the two paths, e.g. after beam splitting only measure N-photon detection, then the interference has N times faster oscillation, $|1 + e^{iN\phi}|^2/4$, which can reach the Heisenberg limit; see an experiment in Fig. 6.

Fisher information: classical and quantum. For this part, we refer the readers to Ref. [33].

Classical part and Cramer-Rao bound. The goal of this discussion is to estimate an unknown parameter θ and to estimate the uncertainty. Given a θ, the measurement that gives some value x is assumed to follow a probability distribution $f(x|\theta)$. From a given reading x_1, then we estimate the value of θ to be $t(x_1)$. With sufficient statistics, we obtain the exact value

$$\langle t \rangle_\theta = \int dx f(x|\theta) t(x) = \theta.$$

The Fisher information is to quantify how the log likelihood function varies with the unknown parameter,

$$I(\theta) = \int dx f(x|\theta) \left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 = \left\langle \left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 \right\rangle_\theta .$$

The so-called Cramer-Rao bound on the variance of the estimation t for n independent samples or experiments is given via the Fisher information,

$$\Delta^2 t \geq \frac{1}{n I(\theta)}.$$

Proof.
First, let us consider $n_{\text{sample}} = 1$ and introduce the covariance between two functions $g(x)$ and $h(x)$:

$$\text{Cov}[g, h] \equiv \langle (g - \langle g \rangle)(h - \langle h \rangle) \rangle.$$

Specializing this to the conditions (i) $t(x)$ whose average is $\langle t \rangle = \theta$ and (ii) $\partial L(x|\theta)/\partial\theta$ [with $L(x|\theta) \equiv \log f(x|\theta)$] whose average is 0:

$$\int dx f(x|\theta) \frac{\partial}{\partial \theta} L(x|\theta) = \int dx \frac{\partial}{\partial \theta} f(x|\theta) = \frac{\partial}{\partial \theta}(1) = 0,$$

we have

$$\text{Cov}[t(x), \frac{\partial}{\partial \theta} L(x|\theta)] \equiv \langle (t(x) - \theta)(\frac{\partial}{\partial \theta} L(x|\theta) - 0) \rangle = \langle t(x) \frac{\partial}{\partial \theta} L(x|\theta) \rangle = \int dx\ t(x) \frac{\partial}{\partial \theta} f(x|\theta) = \frac{\partial}{\partial \theta} \int dx\ t(x) f(x|\theta) = 1.$$

Then we will use the Cauchy-Schwarz inequality,

$$\Delta^2 g\, \Delta^2 h = \langle (g - \langle g \rangle)^2 \rangle \, \langle (h - \langle h \rangle)^2 \rangle \geq |\langle (g - \langle g \rangle)(h - \langle h \rangle) \rangle|^2,$$

noting that

$$\Delta^2 \frac{\partial}{\partial \theta} L = I(\theta),$$

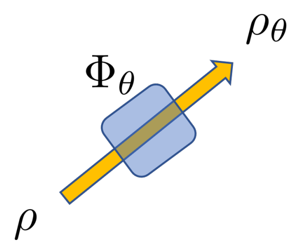

FIG. 7. Illustration of quantum sensing: estimating a phase.

we obtain (for n samples)

$$\Delta^2 t \geq \frac{1}{n_{\rm sample} I(\theta)}.$$

Quantum part. In this case we use a quantum state ρ that is under some quantum operation Φ_θ, as illustrated in Fig. 7, i.e.

$$\rho_\theta = \Phi_\theta(\rho).$$

We want to estimate the parameter θ and calculate the lower bound on its variance. How do we generalize the classical case? How do we generalize $\partial L(x|\theta)/\partial\theta$ (where $L(x|\theta) \equiv \log f(x|\theta)$)? Intuitively $f(x|\theta) \sim \rho_\theta$ (which is defined more precisely being measured w.r.t. an operator Π_x), then

$$\frac{\partial}{\partial\theta}\log f(x|\theta) \sim \frac{\partial\rho_\theta}{\partial\theta}/\rho_\theta \sim D_\theta.$$

Thus, we expect

$$I(\theta) = \left\langle \left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2 \right\rangle_\theta \Longrightarrow \mathrm{Tr}\big(\rho_\theta D_\theta^2\big).$$

It turns out that D_θ is defined more precisely via

$$\frac{\partial}{\partial\theta}\rho_\theta = \{\rho_\theta, D_\theta\}/2 =: A,$$

where we also define a quantity A and use the notation of the anticommutator $\{P, Q\} \equiv PQ + QP$.

The Quantum Fisher information is thus

$$I_Q(\rho,\theta) = \mathrm{Tr}(\rho_\theta D_\theta^2).$$

From this, we expect the corresponding quantum Cramer-Rao bound to be

$$\Delta^2 t \geq \frac{1}{n_{\rm sample} I_Q(\rho,\theta)}.$$

We will not show the proof, but we note that $f(x|\theta) \equiv \mathrm{Tr}(\rho_\theta \Pi_x)$ and the proof should proceed as the classical one.

Unitary process. Let us now consider Φ_θ to be a unitary process,

$$\rho_\theta = U(\theta)\rho U^\dagger(\theta), \quad U(\theta) \equiv e^{-i\theta\hat{G}}.$$

The quantum Fisher information, by definition, is

$$I_Q(\rho,\theta) = \mathrm{Tr}(\rho_\theta D_\theta^2).$$

From the definition of A, we have

$$A_{ij} = \langle i|\partial\rho_\theta/\partial\theta|j\rangle = \langle i|\frac{1}{2}(\rho_\theta D_\theta + D_\theta\rho_\theta)|j\rangle,$$

and we can choose specifically $|i\rangle$'s to the eigenbasis, $\rho_\theta|j\rangle = \lambda_j|j\rangle$. Thus

$$A_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)D_{ij},$$

which leads to

$$D_\theta = 2\sum_{i,j} \frac{\langle i|\partial\rho_\theta/\partial\theta|j\rangle}{\lambda_i + \lambda_j}|i\rangle\langle j|.$$

In our case study where Φ_θ is unitary, the eigenspectrum of ρ_θ is the same as that of ρ, and we have that the Quantum Fisher information is independent of θ, (noting $\partial\rho_\theta/\partial\theta = -i[\hat{G}, \rho_\theta]$)

$$I_Q(\rho, \theta, \hat{G}) = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}|\langle i_0|G|j_0\rangle|^2, \quad \rho = \sum_i \lambda_i |i_0\rangle\langle i_0|,$$

where the subscript 0 is used to denote the original eigenbasis. The detailed derivation is left as an exercise.

$$I_Q(|\psi\rangle, \theta, \hat{G}) = 4(\langle\psi|\hat{G}^2|\psi\rangle - \langle\psi|\hat{G}|\psi\rangle^2) = 4(\Delta G)^2$$

$$\Delta^2 t \geq \frac{1}{n_{\rm sample} I_Q(\rho, \theta)}$$

Example. Coherent state: We have seen the coherenct state in an earlier unit, which is a good description of lasers.

$$\rho = |\alpha\rangle\langle\alpha|, \ \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

We want to see how good it is for the estimation of a unknown phase. Plugging it into the Fisher information, we find that $I_Q = |\alpha|^2$ is proportional to the average number of photons.

Fock states: these are fixed photon-number states,

$$\rho = |n\rangle\langle n|, \ \hat{a}|n\rangle = \sqrt{n}|n\rangle.$$

It is easy to see that

$$I_Q = 0,$$

and hence these states are not useful for estimating the phase, as it will gives very large uncertainty, $\Delta t \sim 1/\sqrt{I_Q}$.

Superposition: What if we consider a superposition of two number states, say, the vaccum and $|n\rangle$?

$$\rho = |\psi\rangle\langle\psi|, \ |\psi\rangle = (|0\rangle + |n\rangle)/\sqrt{2}$$

It is straightforward to show that

$$I_Q = n^2,$$

which achieves the Heisenberg limit, as $\Delta t \sim 1/n$.

When the input is pure and there are multiple parameters. We have seen that for a single parameter case,

$$|\psi_\theta\rangle = U(\theta)|\psi\rangle, \ |\partial_\theta\psi_\theta\rangle \equiv \frac{\partial}{\partial\theta}|\psi_\theta\rangle,$$

the quantum Fisher information for a pure state has a much simpler expression:

$$I_Q(|\psi\rangle, \theta) = 4(\langle\partial_\theta\psi_\theta|\partial_\theta\psi_\theta\rangle - |\langle\psi_\theta|\partial_\theta\psi_\theta\rangle|^2).$$

We can generalize this to multiple parameters and define the following quantum Fisher 'matrix':

$$(I_Q)_{i,j} = 4{\rm Re}(\langle\partial_{\theta_i}\psi_\theta|\partial_{\theta_j}\psi_\theta\rangle - \langle\psi_\theta|\partial_{\theta_i}\psi_\theta\rangle\langle\partial_{\theta_j}\psi_\theta|\psi_\theta\rangle).$$

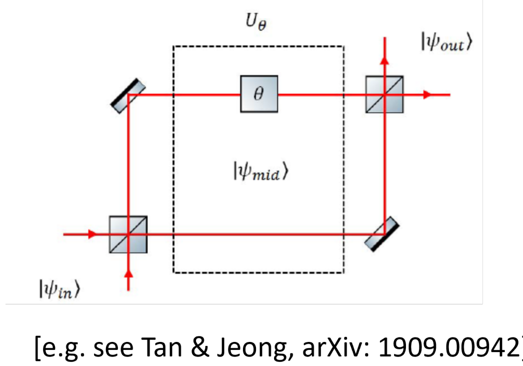

FIG. 8. Illustration of an interferometer and estimation of phases; Tan and Jung, arXiv:1909.00942 [33].

We note that the quantum Fisher 'matrix' (or 'metric') was recently used in the gradient decent approach for the VQE algorithm. See e.g. "Simultaneous Perturbation Stochastic Approximation of the Quantum Fisher Information," by Julien Gacon, Christa Zoufal, Giuseppe Carleo, Stefan Woerner, Quantum 5, 567 (2021) at `https://quantum-journal.org/papers/q-2021-10-20-567/` [34] and "Fisher Information in Noisy Intermediate-Scale Quantum Applications," by Johannes Jakob Meyer, published in Quantum 5, 539 (2021), `https://arxiv.org/pdf/2103.15191.pdf` [35], with a perspective article by K. Bharti at `https://quantum-journal.org/views/qv-2021-10-06-61/` [36].

Example. See Fig. 8 for the illustration.

$$\hat{G} = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2$$

$$I_Q(|\psi\rangle, \theta, \hat{G}) = 4(\langle\psi|\hat{G}^2|\psi\rangle - \langle\psi|\hat{G}|\psi\rangle^2) = 4(\Delta G)^2.$$

There are two modes, and $\hat{G}$ introduces a phase difference between the two modes. We will send separately coherent states into these,

$$|\alpha\rangle \otimes |\beta\rangle, \;\; \hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \hat{b}|\beta\rangle = \beta|\beta\rangle.$$

We find that the use of coherent states (in two modes) leads to the shot-noise limit,

$$I_Q = |\alpha|^2 + |\beta|^2.$$

However, if we send in a N00N state:

$$|\text{N00N}\rangle \equiv (|N_A, 0_B\rangle + |0_A, N_B\rangle)/\sqrt{2},$$

we find that

$$I_Q = N^2,$$

showing that we can achieve the Heisenberg limit.

V. CONCLUDING REMARKS

In this unit, we have discussed quantum simulations and quantum sensing and metrology.

It is a good time to check whether you have achieved the following Learning Outcomes:
After this Unit, You'll be able to get some glimpses to quantum simulations and sensing and metrology and to explain them.

Suggested reading: Richard Feynman's paper "Simulating Physics with Computers", Int. J. Theor. Physics, vol 21, no.6/7, p467-488 (1982) [2].

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