Unit 2: From Foundation to Science-Fiction Teleportation

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In this unit, we discuss the CHSH-Bell inequality and PR Box that are basic topics in quantum foundation. This allows us to understand the difference between classical local theory, quantum mechanics, and general no-signalling theory. We then move on to discuss variable protocols of quantum information processing, such as quantum teleportation, entanglement swapping and superdense coding.

Learning outcomes: (1) You'll be able to understand what CHSH-Bell inequality is about. (2) You'll be able to understand basic and important protocols of quantum information processing.

I. INTRODUCTION

Quantum entangled states have correlations stronger than classical states. The earliest discussion on such correlation originated in a paper by Einstein, Podolsky and Rosen on "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [1] They wrote down what we now call an entangled state of two particles,

$$\Psi(x_1,x_2)=\int_{-\infty}^{\infty}\varphi_x(x_2)v_x(x_1)dx=\int_{-\infty}^{\infty}e^{(2\pi i/h)(x_1-x_2+x_0)p}dp,$$

and use it to discuss the element of physical reality and completeness of quantum theory, concluding that there was something incompatible in the theory. A more moden example is a singlet state of two spins, which is what Jonh Bell used for his famous inequality [2] that shows different results between quantum and classical theories. What we will use to illustrate the discrepancy between quantum and classical theories is the inequality constructed by Clauser, Horne, Shimony and Holt (CHSH) [3]. Experimental violation of Bell's inequalities was the subject of 2022 Nobel Prize in Physics (see `https://www.nobelprize.org/prizes/physics/2022/summary/`), which was awarded to Alain Aspect, John F. Clauser and Anton Zeilinger.

It turns out that entangled states such as the singlet and other so-called Bell states can be a useful resource for a few information processing tasks, such as quantum teleportation, superdense coding, quantum key distribution, etc., a paradigm shift which we will study in this unit.

II. CHSH-BELL INEQUALITY

We have seen measurement of observables X, Y, Z or any one-qubit operator $\vec{r}\cdot\vec{\sigma}$, where $\vec{\sigma}\equiv(X,Y,Z)$ and $|\vec{r}|=1$, gives an eigenvalue randomly, which is ±1 in this case. For variables (e.g. a, a', b, and b' below) that can be only $+1$ or -1, there is an interesting equality:

$$ab+ab'+a'b-a'b'=a(b+b')+a'(b-b')=\pm2,$$

which you can easily verify and convince yourself. (Such a binary number +1 or -1 can be obtained from an experiment: e.g., if one measures the spin in the Stern-Garlach setup, the one with up is assigned +1 while the one with down is assigned -1.) Thus, for any probability distribution $p(a,a',b,b')$ we have (using $\mathbf{E}$ to denote expectation),

$$-2\le\mathbf{E}(ab+ab'+a'b-a'b')\equiv\sum_{a,a',b,b'}\rho(a,a',b,b')(ab+ab'+a'b-a'b')\le2,$$

where $\rho(a,a',b,b')$ is some probability distribution used to perform the average. In the context of measuring two choices of observables at two locations A: a & a', location B: b & b', we have the so-called Clauser-Horne-Shimony-Holt (CHSH) inequality:

$$-2\le\mathbf{E}(a,b)+\mathbf{E}(a,b')+\mathbf{E}(a',b)-\mathbf{E}(a',b')\le2.$$

Each $\mathbf{E}$ can be measured experimentally: for a given pair of axes, it is the probability sum of parallel outcomes minus the probability sum of anti-parallel outcomes. The inequality is satisified by any classical local theories.

FIG. 1. Illustration of a source emitting pairs of photons. The circles with a cross are used to indicate the measurement axes.

CHSH generalized John Bell's idea (his original Bell inequality) and made their inequality easier to test experimentally. As we show see below, that quantum mechanics can violate this inequality. The setting that can be used to test this and violate the inequality is a source that emits, e.g., a pair of photons that need to be entangled; see. Fig. 1. The choice of measurement axis (a or a') at A or (b or b') at B cannot affect the outcome of the other side, otherwise, there could be superluminal communication. Nevertheless, from a classical mechanics' perspective, outcomes can be correlated and should be described by some unknown-to-us distribution (depending on some hidden variable λ). This is also called the "local hidden variable" (LHV) theory. The expectation of the measured outcomes is thus an average over the hidden variable,

$$E_L(a,b) \equiv \int d\lambda\, \rho(\lambda) A(a,\lambda) B(b,\lambda),$$

where we have used $A(a,\lambda)$ and $B(b,\lambda)$ to denote the respective measured values ± 1 that can in principle depend on the hidden variable. (This is the probability sum of parallel outcomes minus the probability sum of anti-parallel outcomes.) The subscript L is used to denote the local hidden theory. Specifically, where $A(a,\lambda) = \pm 1$ and $B(b,\lambda) = \pm 1$ can be thought of predetermined results for the measurement settings a at A and b at B depending on the local hidden variable λ; $\rho(\lambda)$ is its distribution.

Locality requires that the outcome $A(a,\lambda)$ does not depend on setting b and that of $B(b,\lambda)$ does not depend on setting a. Later we will also see that a nonlocal theory can maximally violate the inequality.

By averaging over the local hidden variable, we still have the same inequality,

$$|E_L(a,b) + E_L(a,b') + E_L(a',b) - E_L(a',b')| \le 2. \tag{1}$$

The inequality is violated by Quantum Mechanics. To be specific, the operators to be measured are the Pauli operators $\vec{\sigma}$. Let $E_Q(a,b) \equiv \langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle$ denote expectation of repeated measurement along axes of unit vectors $\vec{a}$ and $\vec{b}$, respectively. Define

$$2B \equiv \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b} + \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b'} + \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b} - \vec{\sigma}\cdot\vec{a'}\otimes\vec{\sigma}\cdot\vec{b'}.$$

We consider an experimental setup in Fig. 2. The claim is that for a singlet state $|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, the maximal expectation value of $2B$ over all measurement settings is

$$\max_{a,a',b,b'} |\langle\psi|2B|\psi\rangle| = 2\sqrt{2},$$

which can be achieved for the setting $\theta_a = \pi/2$, $\theta'_a = 0$, $\theta_b = \pi/4$, and $\theta'_b = 3\pi/4$, where the angles are measured from the z-axis in the $z-x$ plane, as shown in the figure.

To verify this, we first note an identity for the single state $|\psi\rangle$, which you can prove by direct calculation using vectors and matrices or by properties of the singlet state under Pauli matrices (this is given as an **exercise**),

$$\langle\psi|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}|\psi\rangle = -\vec{a}\cdot\vec{b}.$$

With this, we can easily verify the violation $2\sqrt{2} > 2$ with this measurement setting, as $\vec{a}\cdot\vec{b} = \cos(\theta_a - \theta_b)$ and it is $\pm 1/\sqrt{2}$ for the axes chosen. The maximal bound $2\sqrt{2}$ is the so-called Tsirelson bound [4]. Deriving maximal violation and measurement settings for an arbitrary state is a math problem. But fortunately this was already solved; see Horodecki et al. [5, 6].

We also note that the other three 'Bell' states also violate the inequality to the same degree,

$$|\Phi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle), \quad |\Psi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle).$$

Entanglement is also an important part and discussion in the black hole information paradox; see, e.g., Ref. [7]. An exercise that readers can do is to figure out the settings of these other cases that achieve the maximal violation, given the case for the singlet.

Experiments. One of the earlier experiments was done by Aspect, Granger and Roger in 1982 [8]. There have been many experiments and there are several so-called loopholes that need to be closed in order to claim a complete victory of quantum mechanics over classical mechanics. I refer to a useful review article on "Bell's Theorem" in Stanford Encyclopedia of Philosophy at `https://plato.stanford.edu/entries/bell-theorem/`.

Bell's inequality violation and Ekert's quantum key distribution. We will discuss quantum key distribution in a later unit. But with the setup for violating Bell's inequality shown in Fig. 2, we can add two additional axes (shown dashed-dotted), so that there are two pairs of axes that match on both sides. Then the two obsevers, Alice and Bob, meaure their particles with three possible axes randomly chosen on their side. They compare these axes afterwards. If their axes match, then their results must be anti-correlated (assuming using the singlet state $|\Psi^-\rangle$) and can be used to establish a common long sequence of bit string (of course after flipping the anticorrelated results). The rest can be used to calculate the violation of the CHSH-Bell inequality. The larger the violation, the better the security. This was first proposed by Ekert [9], which turns the original subject of philosophical debate into a technologically useful advantage.

III. A TWO-PLAYER GAME AND THE POPESCU-ROHRLICH BOX

A game. Here, consider a related inequality and a game between two players moderated by a referee. The game is played as follows. First, the referee gives two bits x and y separately to Alice and Bob (without them knowing the other's bit). Then, Alice and Bob have to each produce a bit respectively (a and b) without communication. They win the game if

$$x \,\text{AND}\, y = a \oplus b.$$

There are 8 winning cases out of a total of 16 possibilities. To play this game, Alice and Bob need to establish a strategy beforehand and it is likely to follow some probablistic distribution. The conditional probability $P(a, b|x, y)$ that given x and y, a and b is produced is referred to as a "Box". The question we are interested in is how to maximize the probability of winning, which surely depends on the models for P.

It turns out that: (1) For classical no-signaling theory, the maximum winning probability is 3/4. We won't be able to prove the optimality but this answer can be understood by a particular stragety Alice and Bob take: they always output 0. In four possible combinations of (x, y), three of them give $x \,\text{AND}\, y = 0$ and one gives 1.

(2) For quantum mechanics, the maximum winning probability is $(2 + \sqrt{2})/4 \approx 0.8535$. The $\sqrt{2}$ is related to the $2\sqrt{2}$ in the maximal violation of the CHSH-Bell inequality. The connection basically comes from the following. We regard x and $y \in \{0, 1\}$ as representing two different measurement settings and a and $b \in \{0, 1\}$ as representing two different outcomes (i.e +1 and -1). Then, we can relate the probability of winning P_{win} to the expectation of B via

$$P_{\text{win}} = \frac{1}{2} + \frac{\langle B \rangle}{4}. \tag{2}$$

From the maximality of CHSH-inequality violation, we have

$$\max P_{\text{win}} = \frac{1}{2} + \frac{\sqrt{2}}{4}.$$

That this is the maximum for quantum mechanics was proven by Tirelson [4].
Exercise. Prove Equation (2).

The PR Box. It is important that any box $P(a, b|x, y)$ according to any strategy cannot violate the no-signaling condition: for example, $P(0, 0|0, 0)+P(0, 1|0, 0) = P(0, 0|0, 1)+P(0, 1|0, 1)$, i.e. the 'marginal' probability by summing over the output of either Alice or Bob should not depend on the setting x or y on the other side.

Popescu and Rohrlich proposed a 'box' [10] that always achieves $x \,\text{AND}\, y = a \oplus b$: $P(0, 0|0, 0) = P(1, 1|0, 0) = P(0, 0|0, 1) = P(1, 1|0, 1) = P(0, 0|1, 0) = P(1, 1|1, 0) = P(0, 1|1, 1) = P(1, 0|1, 1) = 1/2$ (all other combinations are zero). It turns out that this corresponds to simply setting all the conditional probabilities corresponding to non-winning cases to zero and all the winning ones to 1/2. This is called the PR Box and luckily does not violate the no-signaling condition, and it allows to win the above game with 100% probability. The PR Box allows much higher violation of the CHSH inequality, giving $|2B| = 4$, and thus it has correlation stronger than quantum mechanics. For those interested in delving deeper, I highly recommend a review article by Popescu [11].

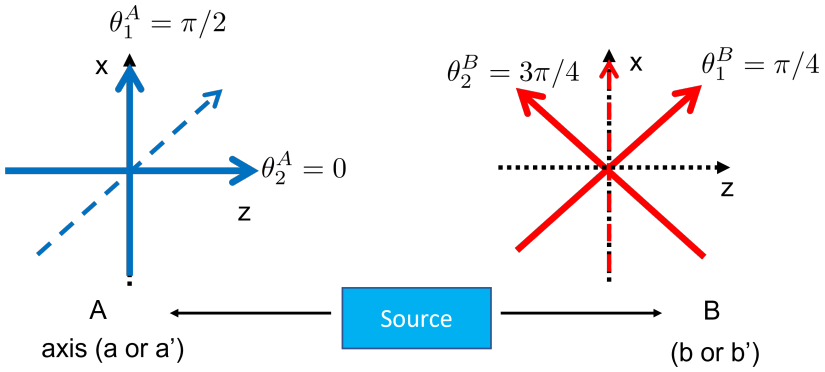


FIG. 2. Illustration of the measurement setup for testing the CHSH inequality for the quantum state being $|\Psi^-\rangle$. Optimal axes for other Bell states can be obtained by suitably rotating the axes shown here. The dash-dotted axes are added for the purpose of quantum key distribution using Ekert's protocol.

IV. GHZ STATE: VIOLATION AT A SINGLE SHOT

We can generalize the Bell state $|\Phi^+\rangle$ to three particles and arrive at the Greenberger-Horne-Zeilinger state [12],

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Now consider four commuting observables: (i) $X \otimes X \otimes X$, (ii) $Y \otimes Y \otimes X$, (iii) $Y \otimes X \otimes Y$, (iv) $X \otimes Y \otimes Y$. You can verify that they mutually commute as any pair differs by two locations. You can also verify the following relations,

$$X \otimes X \otimes X|\text{GHZ}\rangle = (+1)|\text{GHZ}\rangle, \quad (3)$$
$$Y \otimes Y \otimes X|\text{GHZ}\rangle = (-1)|\text{GHZ}\rangle, \quad (4)$$
$$Y \otimes X \otimes Y|\text{GHZ}\rangle = (-1)|\text{GHZ}\rangle, \quad (5)$$
$$X \otimes Y \otimes Y|\text{GHZ}\rangle = (-1)|\text{GHZ}\rangle. \quad (6)$$

We can multiple all four sets of operators and both left and right hand sides give -1.

How do we explain the above four equations in terms of a classical theory? For a classical local theory, one attributes these to local properties (with classical variables x_i's and y_i's):

$$x_1x_2x_3 = +1,\ y_1y_2x_3 = -1,\ y_1x_2y_3 = -1,\ x_1y_2y_3 = -1,$$

where $x_i, y_i = \pm 1$. But this gives contradiction when we multiply all four equalities together: $1 = -1$! (experiments e.g. by Dirk Bouwmeester and collaborators showed that QM is correct [13, 14]).

We note that this violation can also be formulated as a three-player game. There are many subjects in quantum foundation that we do not have time to cover, such as Mermin's magic square, Kochen-Specker theorem, and whether the measurement postulate is necessary or not [15].

V. QUANTUM ENTANGLED STATES ARE USEFUL

We shall discuss several quantum information processing protocols proposed earlier in the development which use simple entangled states such as Bell states and their generalization. These include quantum teleportation, entanglement swapping, gate teleporation, superdense coding, remote state preparation, and superdense teleportation. In my opinion, quantum teleporation is the most unexpected among these.

A. Quantum teleportation

As said earlier, one of the most incredible tasks that an entangled pair allows is quantum teleportation invented by Bennett and collaborators [16]. For illustration, we use the state $|\Phi^+\rangle$ to explain this. Suppose we have an arbitrary

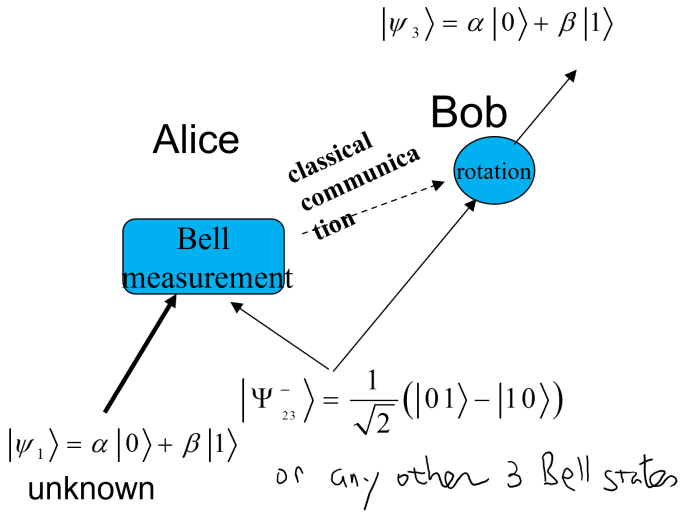


FIG. 3. Illustration of quantum teleportation. In this diagram, the unknown state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and the shared entanglement is in the form of the singlet $|\Psi^-\rangle$. The other Bell states can also be used.

(and unknown) state $|\psi\rangle_1$ of particle 1 at A, who shares the entanglement with B via $|\Phi^+\rangle_{23}$. The numbers in the subscripts indicate which particles they are. Now let

$$|\psi\rangle_1 = a|0\rangle + b|1\rangle,\ |\psi\rangle_1 \otimes |\Phi^+\rangle_{23} = \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle)_1 \otimes (|00\rangle + |11\rangle)_{23}.$$

Now we will do some manipulation on the state, and expand the right-hand side:

$$\begin{aligned}
\text{r.h.s.} &= \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle) && (7)\\
&= \frac{1}{2}\left\{a(|\Phi^+\rangle + |\Phi^-\rangle)\otimes|0\rangle + a(|\Psi^+\rangle + |\Psi^-\rangle)\otimes|1\rangle + b(|\Psi^+\rangle - |\Psi^-\rangle)\otimes|0\rangle + b(|\Phi^+\rangle - |\Phi^-\rangle)\otimes|1\rangle\right\} && (8)\\
&= \frac{1}{2}\left\{|\Phi^+\rangle\otimes\underbrace{(a|0\rangle + b|1\rangle)}_{|\psi\rangle} + |\Phi^-\rangle\otimes\underbrace{(a|0\rangle - b|1\rangle)}_{Z|\psi\rangle} + |\Psi^+\rangle\otimes\underbrace{(a|1\rangle + b|0\rangle)}_{X|\psi\rangle} + |\Psi^-\rangle\otimes\underbrace{(a|1\rangle - b|0\rangle)}_{iY|\psi\rangle}\right\}. && (9)
\end{aligned}$$

The unknown information $a\&b$ is preserved in the third particle, but depending on the outcome of the 'Bell-state' measurement (in the basis of $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$, there is an additional operator acting on $|\psi\rangle$, which can be undone if the measurement outcome is known.

Summarizing what the two parties need to do after Alice measures her particles 1 and 2, there four possible outcomes, Alice informs Bob: (1) it is $|\Phi^+\rangle$, please apply identity (nothing); (2) it is $|\Phi^-\rangle$, please apply Z to particle 3; (3) it is $|\Psi^+\rangle$, please apply X to particle 3; (4) it is $|\Psi^-\rangle$, please apply $-iY$ to particle 3. At the end, Bob recovers the state $|\psi\rangle$ at particle 3. (We note that in case (4), $-iY$ formally cancels out iY. But applying Y also recovers $|\psi\rangle$ up to an irrelevant global phase i.) Moreover, the probability of obtaining any of the four outcomes is $(1/2)^2 = 1/4$, according to our measurement postulate discussed in Unit 1. The setup of quantum teleportation is illustrated in Fig. 3. Quantum teleportation was invented by Bennett and collaborators in early 1990s.

Exercise. (a). We explain the teleportation using the resource state $|\Phi^+\rangle$. In Fig. 3, we illustrate the process using $|\Psi^-\rangle$. Work out the correction operator needed for all four measurement outcomes. (b). In the original paper by Bennett et al. [16], they already worked out the qudit (d-level) case. What are the possible two-qudit entangled resource states and correction operators?

For the experiments using entangled photons, please see Bouwmeester et al. '97 [17] and Pan et al. '03 [18]. The drawback in this system is that the Bell-state measurement cannot be done perfectly, only two of the four Bell states can be distinguished (if no ancillary photons are added). The success rate is thus only 50% [19, 20]. With single-photon ancillas, the scheme can be boosted to at least 3/4 success rate and may possibly be pushed to 100% [21]. There was a recent experiment that demonstrated about 58% of success, beyond the no-ancilla limit [22]. (This is an important practical issue mostly glossed over in textbooks.)

An alternative graphical approach for the teleportation was given in the lecture. Details to be supplied later (maybe in an Appendix).

B. Gate teleportation

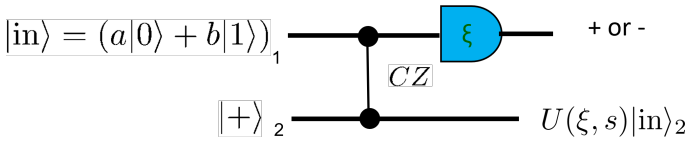


FIG. 4. Illustration of the gate teleportation circuit. CZ indicates the Controlled-Z or gate and ξ is the measurement angle, defined in the main text. Although we have not discussed extensively quantum circuits, we have seen them in the previous chapter with simple algorithms.

[The following material in this section was taken from my review paper: "Quantum spin systems for measurement-based quantum computation" [23].]

Consider an arbitrary qubit state $|\psi\rangle_1 = a|0\rangle_1 + b|1\rangle_1$, where the subscript 1 is used to label the qubit, and another qubit in the state $|+\rangle_2 = (|0\rangle_2 + |1\rangle_2)/\sqrt{2}$ state. Note that $|0\rangle$ and $|1\rangle$ are the +1 and -1 eigenstates, respectively, of the Pauli Z matrix $Z = \sigma^z$. In the following, the normalization, such as $1/\sqrt{2}$, may be dropped for ease of notation. Imagine we have a Controlled-Z gate $\text{CZ}_{mn} \equiv |0\rangle\langle 0|_m \otimes \mathbb{1}_n + |1\rangle\langle 1|_m \otimes Z_n$. Note that the CZ gate is actually symmetric, $\text{CZ}_{mn} = \text{CZ}_{nm}$, as the nontrivial action on the two qubits is only a phase shift: $|11\rangle \to -|11\rangle$. This can come from, e.g., an Ising interaction in the presence of an external field.

Applying the CZ gate to the two qubits: $\big(a|0\rangle_1 + b|1\rangle_1\big)|+\rangle_2 \to |\Psi\rangle_{12} = a|0+\rangle + b|1-\rangle$, wehre $|-\rangle \equiv (|0\rangle_2 - |1\rangle_2)/\sqrt{2}$; an entanglement is thus created between qubits 1 and 2. Imagine we perform a projective measurement on the first qubit, described by the observable $\hat{O}(\xi) = \cos\xi\, X + \sin\xi\, Y$ (recall the measurement postulate in Unit 1), where $X = \sigma^x$ and $Y = \sigma^y$ are the Pauli X and Y matrices, respectively. An equivalent description of the measurement is the eigenstates of the observable $|\pm\xi\rangle \equiv (|0\rangle \pm e^{i\xi}|1\rangle)/\sqrt{2}$, with eigenvalues $\pm 1 = (-1)^s$ (or equivalently a binary variable $s = 0, 1$) to describe the measurement outcome.

Depending on the measurement outcome s on the first qubit, the second qubit is projected to (the derivation is left as an exercise or see the lecture)

$$|\psi'\rangle_2 = \langle \pm\xi|\Psi\rangle_{12} \sim H\, e^{i\xi Z/2}\, Z^s (a|0\rangle_2 + b|1\rangle_2), \tag{10}$$

where an overall phase factor is omitted. Such a procedure of (1) entangling an arbitrary input qubit $|\text{in}\rangle$ with a fixed $|+\rangle$, followed by (2) measuring the first qubit in $|\pm\xi\rangle$ basis, results in the quantum information $|\text{in}\rangle$ teleported to the second qubit, with an additional outcome-dependent unitary gate $U(\xi, s) = H\, e^{i\xi Z/2}\, Z^s$, where H is not a Hamiltonian but the so-called Hadamard gate (see Unit 1)

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{11}$$

This is referred to as the gate teleportation, schematically shown in Fig. 4. We shall see later in this course that this gate teleportation can be used to understand an alternative framework of quantum computation: the measurement-based quantum computation [24].

C. Entanglement swapping

Here we discuss the entanglement swapping [25], which is a direct application of quantum teleportation. The consequence is that two particles that were not previously entanglement can be made to become entangled via Bell measurement on their respective partner particles. From the point of view of teleportation, the entanglement between, e.g. A and B_1, is teleported to that between A and C by the procedure of teleporting B_1 to C (via the entangled pair shared between B_2 and C). That is, C inherits the prior entanglement of B_1 with A.

As illustrated in Fig. 5, imagine that Alice and Bob share an entangled pair and Bob and Charlie share another entangled pair. By performing the Bell-state measurement on Bob's two particles, Bob 'teleports' his entanglement

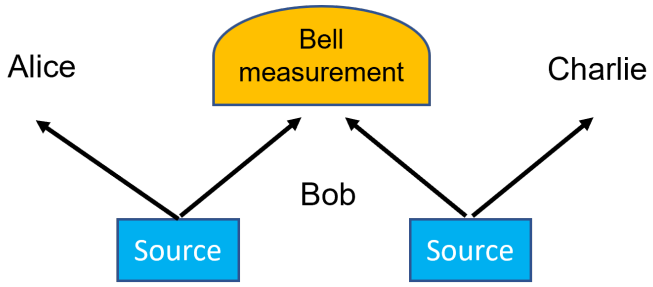


FIG. 5. Illustration of entanglement swapping.

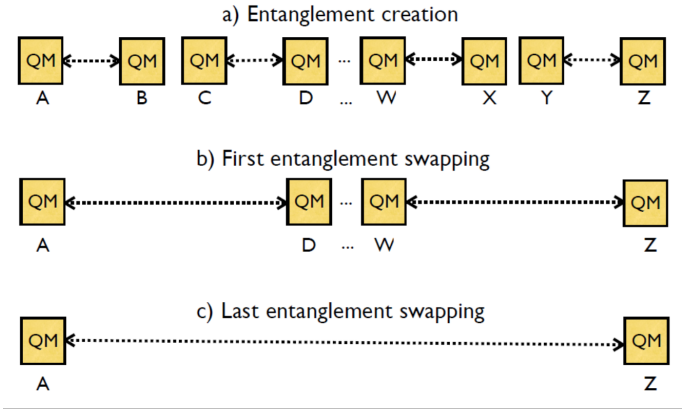


FIG. 6. Illustration of the DLCZ quantum repeater protocol [26]; figure taken from [27].

with Alice to Charlie (or equivalently, Bob 'teleports' his entanglement with Charlie to Alice). This results in shared entanglement between Alice and Charlie.

The entanglement swapping is the basic element to establish entanglement between distant nodes (such as the Duan-Luken-Cirac-Zoller with atomic ensemble quantum memory [26]) if there are many entangled pairs shared along a path of nodes. We will see this in a later unit on quantum communication.

D. Remote State Preparation

A Bell state, such as the singlet, has strong correlation, and measurement outcomes on two sides are correlated. Using this strong quantum correlation and the collapse of wave function by measurement, it is possible to remotely prepare a state [28] (plus some possible additional 'flip' on the particle whose state is remotely prepared). Let us illustrate this with the singlet state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

From its antisymmetry, we can see that for any single qubit state $|\psi\rangle$ and its orthogonal state $|\psi^\perp\rangle$ we use them to as a basis to rewrite the singlet state,

$$|\Psi^-\rangle = e^{i\theta}\frac{1}{\sqrt{2}}(|\psi\rangle \otimes |\psi^\perp\rangle - |\psi^\perp\rangle \otimes |\psi\rangle),$$

where $e^{i\theta}$ is some global phase. If Alice performs measurement on her particle in the basis $\{\psi, \psi^\perp\}$, with probability 1/2, she obtains $\psi^\perp$ and thus prepares Bob's state in $|\psi\rangle$. Similarly with probability 1/2, Alice prepares Bob's state in $|\psi^\perp\rangle$. However, in the latter case, it is in general impossible for Bob to transform from in $|\psi^\perp\rangle$ to $|\psi\rangle$, except for those 'equatorial states',

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle,$$

whose orthogonal counterpart is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\phi}|1\rangle.$$

The operation to 'flip' in this case is Z. The notion of remote state preparation and the first pioneering work was done by Bennett and collaborators.

E. Superdense coding

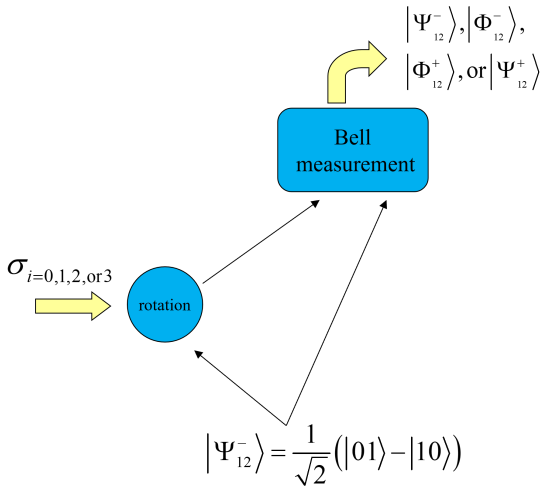


FIG. 7. Illustration of superdense coding.

A classical bit can encode two different messages and when it is transmitted physically, at most one bit of information can be transmitted. However, we will see that by sending a physical quantum bit, at most two bits of information can be transmitted [29]. By now, you probably have the feeling and predict that it required pre-shared entanglement. As illustrated in Fig. 7, Alice can send two bits of classical message to Bob by sending one physical qubit!

It is easy to understand how this is possible and the required math regards local convertibility among the four Bell states, i.e., the conversion can be done on either side by applying an action corresponding to a Pauli matrix. For convenience, we display the four Bell states below,

$$|\Phi^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \ |\Psi^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

We give examples of how to convert them locally,

$$|\Phi^-\rangle = Z \otimes I|\Phi^+\rangle = I \otimes Z|\Phi^+\rangle, \ |\Psi^+\rangle = X \otimes I|\Phi^+\rangle = I \otimes Z|\Phi^+\rangle, \ |\Psi^-\rangle = -iY \otimes I|\Phi^+\rangle = I \otimes (iY)|\Phi^+\rangle.$$

Alice performs one of the four operations $\sigma_{i=0,1,2,3}$ (note $\sigma_0 = I$ identity) on her particle and sends her particle to Bob. Then Bob possesses both particles and if he knows the initial shared entanglement is $|\Phi^+\rangle$, he can perform a Bell measurement and infer what operation Alice has performed using the above equations of the local conversion.

The superdense coding was first performed by Mattle et al. '96 using entangled photons. However, as we have mentioned earlier, only two of the Bell states ($|\Psi^\pm\rangle$) can be distinguished and the other two $|\Phi^\pm\rangle$ cannot be distinguished between themselves, we can encode at most three messages, i.e. $\log_2(3) \approx 1.585$ bits of information. However, using other degrees of photons, such as the orbital angular momentum, 'hyper-entanglement' can be used to achieve encoding of 2 bits of information. This was proposed by Kwiat and collaborators and an experiment to demonstrate this was carried out in Kwiat's group.

F. Superdense teleportation

Here, I describe a protocol called superdense teleportation, which was carried out in a work that I was involved with. This protocol was originally proposed by Herbert Bernstein and the experiment was performed by the group of

Paul Kwiat at the University of Illinois [30]. In this setup, Alice wants to send the following state (with phases ϕ's supplied by Charlie) to Bob,

$$|\psi\rangle_1 = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_j} |j\rangle, \quad \text{with } \phi_0 = 0,$$

The shared entangled state between Alice and Bob is

$$|\Phi\rangle_{12} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j,j\rangle,$$

a d-level qutrit pair of entanglement, which you may have seen in the exercise of quantum teleportation. Charlie applies the phase shift $U_\phi = \sum_{j=0}^{d-1} e^{i\phi_j} |j\rangle\langle j|$ to particle 1 (of Alice) and measure it in the basis (a Fourier basis) defined by

$$|\tilde{k}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i2\pi jk/d} |j\rangle.$$

Upon obtaining $\tilde{k}$, Alice projects Bob's particle to the following state

$$\langle \tilde{k}|_1 |U_\phi \otimes I|\Phi\rangle_{12} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_j} \langle \tilde{k}|_1 \cdot |j,j\rangle = \frac{1}{d} \sum_j e^{i\phi_j - i2\pi jk/d} |j\rangle.$$

Alice informs Bob of outcome k, and Bob applies

$$V_k = \sum_{j=0}^{d-1} e^{i2\pi jk/d} |j\rangle\langle j|$$

to his particle to recover $|\psi\rangle$. In the experiment, the degrees of freedom used are photons' polarization and orbital angular momentum.

[To add other topics such as quantum broadcasting....]

VI. CONCLUDING REMARKS

In this unit, we have discussed the CHSH-Bell inequality, PR Box, and violation of local realistic theory with the GHZ state. These belong to the subfield in quantum foundation. This allows us to understand the difference between classical local realistic theory, quantum mechanics, and general no-signalling theory. After those, we have discussed early protocols of quantum information processing, such as quantum teleportation, entanglement swapping and superdense coding, as well as gate teleportation and superdense teleporation.

It is a good time to check whether you have achieved the following Learning Outcomes:
After this Unit, (1) You'll be able to understand what CHSH-Bell inequality is about. (2) You'll be able to understand basic and important procotols of information processing.

Units 1 and 2 give you a quick approach to the theoretical basics of quantum principles and quantum information processing protocols. In the next unit, we will discuss a few physical systems used in realizing quantum bits and quantum information processing tasks.

Suggested reading: N&C 2.3, 2.6; KLM chapter 5; Qb 3.1-3.5.

Appendix A: Kochen-Specker theorem and Mermin-Peres magic square

There are many topics that could not be included to the lectures. For example, the Kochen-Specker theorem and contexualtiy (which says that it is impossible to assign definite, pre-existing values to all measurable physical

$X \otimes I$	$I \otimes X$	$X \otimes X$
$I \otimes Y$	$Y \otimes I$	$Y \otimes Y$
$X \otimes Y$	$Y \otimes X$	$Z \otimes Z$

TABLE I. Mermin-Peres magic square. Operators in one row/column commute with one another. Product of operators in all rows and first two columns is $I \otimes I$, while that in the first column is $-I \otimes I$. Classical physics assumes objects have definite values even before you measure them, but Mermin's square proves that if you multiply the rows and columns, a classical system creates a logical contradiction 1=-1. In quantum measurements, outcome depends entirely on the full context of what else you measure at the same time. For applications, two players using an entangled quantum state can win the Mermin-Peres game 100% of the time, beating any classical strategy.

properties, i.e., commuting observables, of a quantum system prior to measurement. A quantum particle cannot possess a predetermined value for a property A independent of whether A is measured alongside compatible property B or compatible property C. The measurement choice (the context) directly alters the physical reality). The theorem was proved for the Hilbert space dimension be three or larger; see more detail in Wikipedia `https://en.wikipedia.org/wiki/Kochen-Specker_theorem` and Refs. [31, 32] for experimental tests.

The Mermin-Peres magic square provides the simplest and clearest proof of quantum contextuality and the Kochen-Specker theorem); see the Wikipedia page `https://en.wikipedia.org/wiki/Quantum_pseudo-telepathy`. Table I shows an explicit construction and describes the contradiction, if one insists that each observable has a definite value for some hidden-variable theory.

Given that we have briefly mentioned the Kochen-Specker (KS) theorem and discussed in detail Bell's theorem, it is worth comparing the two. The former targets non-contextual Realism (pre-existing values), while the latter targets local realism (i.e. spooky action at a distance). The former can be tested on a single quantum particle (of dimension 3 or larger), while the latter requires entangled pairs separated at a distance. The former is state independent (contradictions lie in operators), while the latter is state dependent (violation of the inequality depends on specific entangled states used).

Other topics that I would like to cover include Hardy's nonlocality of two particles without inequalities [33] and nonlocality of a single photon [34], etc. [This could be presented by students at the end of the course to supplement the lectures.]

Appendix B: Additional elemental protocols

Some of the content was excerpted from the appendix in my paper with Hiroki Sukeno [35] and it is useful to learn other related protocols.

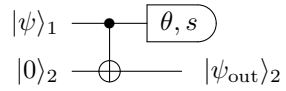
Protocol 1'. From the gate-teleportation protocol, instead of $|+\rangle$ in the second qubit and the CZ gate, we can replace them with the $|0\rangle$ state and the CNOT gate. We have

$$|\psi\rangle_1|0\rangle \xrightarrow{\text{CNOT}} a|00\rangle_{12} + b|11\rangle_{12}. \tag{B1}$$

Measurement on the first qubit in the same $\pm\theta$ base gives

$$(ae^{i\theta/2}|0\rangle \pm be^{-i\theta/2}|1\rangle)_2 = e^{i\frac{\theta}{2}Z}Z^s(a|0\rangle + b|1\rangle)_2, \tag{B2}$$

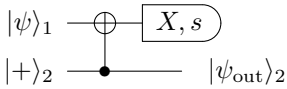
where the resulting teleported gate is $U_{1'}(\theta, s) = e^{iZ\theta/2}Z^s$. This is illustrated as follows,



Protocol 2. In this protocol, we begin qubit 2 in $|+\rangle$ and apply CNOT_{21} followed by the measurement of qubit 1 in the Z basis. measure qubit

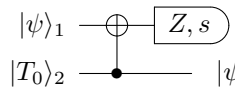
$$\sqrt{2}|\psi\rangle_1|+\rangle_2 \xrightarrow{\text{CNOT}_{21}} (a|0\rangle + b|1\rangle)_1|0\rangle_2 + (a|1\rangle + b|0\rangle)_1|1\rangle_2. \tag{B3}$$

Measurement of qubit 1 in the Z basis yields (labeling the outcome $s = 0, 1$) $X^s(a|0\rangle + b|1\rangle)_2$.



Instead of $|0/1\rangle$, one can also consider a more general basis such as $e^{-i\theta X}|0/1\rangle$.

Protocol 3. In this protocol, instead of $|0\rangle$ or $|+\rangle$ for qubit 2, we will consider a special state $|T_0\rangle \equiv (|0\rangle+e^{i\pi/4}|1\rangle)/\sqrt{2}$. Similar to Protocol 2, we use CNOT_{21}, followed by the Z basis measurement.



$$\sqrt{2}|\psi\rangle_1|T_0\rangle_2 \xrightarrow{\mathrm{CNOT}} (a|0\rangle + b|1\rangle)_1|0\rangle_2 + (a|1\rangle + b|0\rangle)_1 e^{i\frac{\pi}{4}}|1\rangle_2. \tag{B4}$$

When the measurement on qubit 1 gives $s = 0$, the second qubit becomes

$$a|0\rangle_2 + be^{i\frac{\pi}{4}}|1\rangle_2, \tag{B5}$$

which means that qubit 1 is teleported to qubit 2, followed by the non-Clifford T gate. When $s = 1$, the second qubit becomes

$$b|0\rangle_2 + ae^{i\frac{\pi}{4}}|1\rangle_2 = e^{i\frac{\pi}{4}}X(a|0\rangle + be^{-i\frac{\pi}{4}}|1\rangle)_2. \tag{B6}$$

Both cases can be summarized as the gate teleported being $U_3(s) = X^s T^{(-1)^s}$. To achieve the same T gate when $s = 1$, we can apply an X gate, followed by ST, where $S = T^2$. This protocol is useful for magic state injection [36]. We note that the above CNOT_{21} followed by measurement on qubit 1 case can also be regarded as measuring Z_1Z_2. Here, we see that the dynamic circuit, or adaptive gate application, that is dependent on the measurement outcome, is a useful tool for achieving a deterministic effect.

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