Unit 06: Dealing with Errors

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In this unit, we discuss error models, quantum error correction, topological stabilizer codes and topological phases, and error mitigations.

Learning outcomes: You'll be able to understand why quantum information is fragile but quantum correction codes can be used to reduce error rates in logical qubits.

I. INTRODUCTION

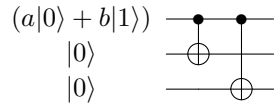
To operate a quantum computer, one needs to actively interact with the quantum device to enact one- and two-qubit gates. One also needs to measure quantum bits to read out the result. It thus impossible to isolate our quantum computer completely from the rest of the world. The interaction with the outside world or the environment can induce errors in our qubits. It was even argued in the beginning of the development that for this reason that quantum computation could not work. For a classical bit, there is only the flip error that flips from 0 to 1 or vice versa with some probability; see Fig. 1a.

But a quantum bit has more errors to occur, e.g. a phase flip: $|0\rangle \to |0\rangle$ but $|1\rangle \to -|1\rangle$. The goal of this unit is to understand how to deal with various types of errors on qubits, so that quantum computation can still persist.

Let us first look at this classical flip channel and how encoding can help to reduce the error rate for the logical bit. We consider the 'redundant' encoding "0"=000, "1"=111; the effect of the bit-flip channel will take 000 to {000: with probability $(1-p)^3$; 001/010/100 with $(1-p)^2p$; 110/101/011: w $(1-p)p^2$; 111: w p^3} and 111 to {111: with probability $(1-p)^3$; 110/101/011 with $(1-p)^2p$; 001/010/100: w. $(1-p)p^2$; 000: w p^3}. We can use the 'majority vote' to recover the original logical bit for some of the cases (e.g. 000 goes to 001/010/100 and the rest 110/101/011 and 111 represents the error. So we have the logical error rate $p' = 3(1-p)p^2 + p^3$. It is obvious from Fig. 2 that $p' < p$ if $p < 1/2$ i.e. the encoding helps. If the original error rate $p > 1/2$, the encoding amplifies the error!

II. QUANTUM ENCODING

Quantum redundant encoding. In the quantum case we have e.g. a quantum bit in an arbitrary pure state $|\psi\rangle = a|0\rangle + b|1\rangle$. One might naively use the redundant encoding as $(a|0\rangle+b|1\rangle)\otimes(a|0\rangle+b|1\rangle)\otimes(a|0\rangle+b|1\rangle)$, as a mapping from $|\psi\rangle|0\rangle|0\rangle$ to this is in general not possible, i.e. no cloning. To overcome this we will use $a|0\rangle+b|1\rangle \to a|000\rangle+b|111\rangle$ and this encoding can be achieved by the circuit



But how do we do the 'majority vote'? We cannot just measure all three qubits in 0/1 as this will collapse the state to a product state, rendering the encoding useless. The solution is to compare them, e.g. by measuring the product of Z_1Z_2 we can compare bits 1&2 to see if they are equal, and simimlary Z_2Z_3 to compare if bit 2 & 3 are equal. By comparing these two results we can figure which qubit has encountered a flip error (of course, this assumes the error rate is small). Then we can apply an X to flip the erroraneous qubit back to the correct state.

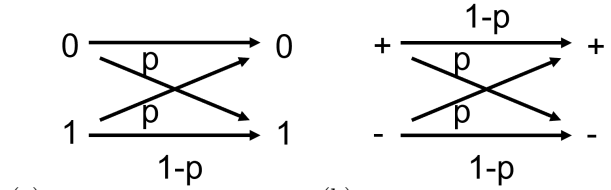
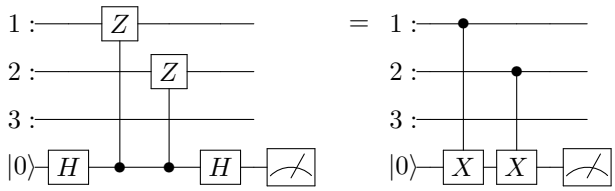


FIG. 1. (a) Bit flip channel. (b) Phase flip channel.

How to measure Z_1Z_2 without measuring separately Z_1 and Z_2? The folloing circuit achieves this,



Here we can introduce the notion of stabilizer operators: Z_1Z_2, Z_2Z_3 and Z_3Z_1 are examples as they 'stablize' the state:

$$Z_1Z_3|\psi\rangle = Z_2Z_3|\psi\rangle = Z_1Z_3|\psi\rangle = |\psi\rangle.$$

Phase flip error. We have more errors than just flipping the bits. Let us consider the phase flip mentioned above: $|0\rangle \to |0\rangle$ but $|1\rangle \to -|1\rangle$, which on any of the qubit will cause the redundant encode to flip from $a|000\rangle + b|111\rangle$ to $a|000\rangle - b|111\rangle$. The above stabilizer measurement cannot detect this error. How can we detect the error?

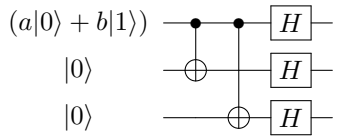
To do that we need to use a different encoding and from the duality between X and Z via the Hadamard gate H, we know that

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle, \qquad |-\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle,$$

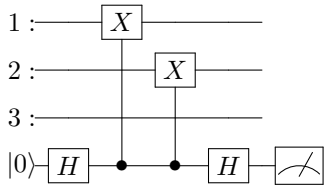
in the $+/-$ basis the phase flip error corresponds to bit flip $+ \leftrightarrow -$. We can thus construct a similar code, dual to the one above,

$$|`0'\rangle \equiv |++\!+\rangle, \ |'1'\rangle \equiv |--\!-\rangle,$$

which can be achieved by



Then the analysis goes exactly the same as in the bit flip channel, except the change $0/1 \to +/-$ and $Z \to X$ and the encoding reduces the phase flip error rate if the original rate $p < 1/2$. The stabilizer operators include X_1X_2, X_2X_3, and X_3X_1. The stabilizer measurement, e.g., X_1X_2 can be done by



Shor's 9-qubit code. How can we deal with both errors? Shor employed both and combined the above to a 9-qubit code. We can regard that the code he constructed is to expand the $|+\rangle$ in the phase flip code each to a three-qubit flip code: $|+\rangle \to |000\rangle + |111\rangle$, and arrive at the following,

$$|"0"\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|"1"\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

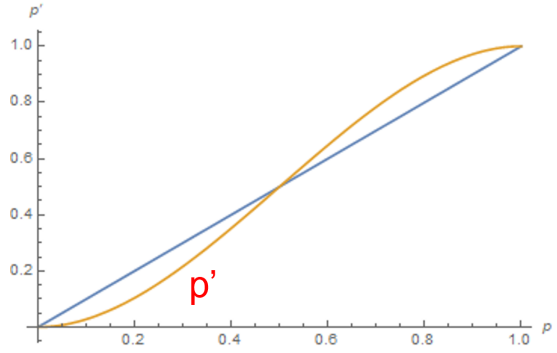
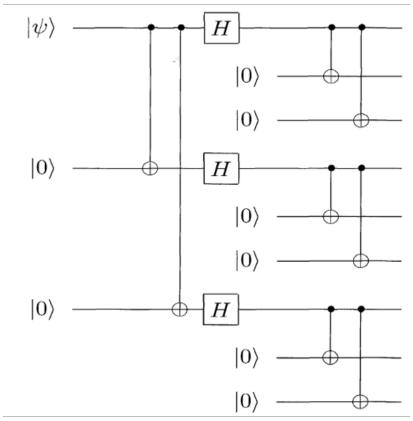


FIG. 2. Illustration of p' as a function of p and the line $p' = p$.

The encoding can be done by the following circuit,



It turns out that this can correct any one-qubit error that is 'any superposition of Pauli errors'.

One can understand this by analyzing the ability to correct bit flip and phase flip of the above two codes that Shor's code inherits. For example, any single bit flip in any of the nine qubits can be detected by running the three-qubit bit flip error detection circuits separately in the three three-qubit blocks: (1,2,3), (4,5,6) and (7,8,9). The phase flip error is slightly different. The previous 3-qubit phase flip code allows determination of which qubit undergoes the phase flip. By generalizing this to the Shor's nine qubits, the flip error can be determined only to within which of the three blocks. The previous stablizer operator X_1X_2 for the three qubits is now generalized to two blocks $(X_1X_2X_3)(X_4X_5X_6)$. The detection circuit now has six CNOT gates (instead of two previously) controlled from the anicillary qubit.

Given that Shor's code can correct single-qubit X and Z errors, by checking and correcting both, we can use it to correct single-qubit Y errors. We will see below that the ability to correct both X and Z errors guarantees the ability to correct the error due to any arbitrary superposition $\alpha X + \beta Z$.

III. ERRORS ARE QUANTUM OPERATIONS

One can consider errors arise from coupling of the system to the environment that causes joint system to evolve unitarily. For example,

$$U(|\psi\rangle|e_0\rangle) = \sum_k (E_k|\psi\rangle)|e_k\rangle,$$

where the operators E_k acts on the system only and we consider the environment's states $|e_j\rangle$'s being orthonormal, $\langle e_j|e_k\rangle = \delta_{jk}$ and $|e_0\rangle$ is some initial state of the environment.

We note that if one insists, one can consider the environment starts with a mixed state ρ_0, which is the reduced density matrix obtained by tracing out another part of some fictitious environment E', $\rho_0 = \mathrm{Tr}_{E'}(|e_0\rangle\langle e_0|)$. We can

$|e_0\rangle$ is the "purification" of ρ_0. As the unitary U preserve the overlap, for two arbitrary system states $|\phi\rangle$ and $|\psi\rangle$,

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \langle\psi|\phi\rangle = \sum_k \langle\psi|E_k^\dagger E_k|\phi\rangle.$$

Given that it holds for arbitrary $|\phi\rangle$ and $|\psi\rangle$, we deduce that

$$\sum_k E_k^\dagger E_k = I,$$

which is a consequence of probability conservation or trace preserving. (Any lossy operation can also be regarded as ignoring certain outcome in some basis. That is if we include all possibilities, there is no loss.)

We begin the discussion using pure states of the system; what if the system is already in a mixed state ρ? How does it evolve under such unitary U coupling with the environment?

$$\rho \otimes |e_0\rangle\langle e_0| \to U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger.$$

Given that we do not have access to the environment, we want to know how the system ρ evolves just in terms of degrees of freedom in the system? The answer is

$$\rho \to \mathrm{Tr}_E[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger] = \sum_k E_k \rho E_k^\dagger.$$

The Tr_E indicates a 'partial trace': tracing over only the environment's degrees of freedom. We have seen the partial trace in Unit 01 already.

Let us examine a few error models below.

Bit flip channel.

$$\rho \to (1-p)\rho + pX\rho\, X^\dagger, \qquad E_1 = \sqrt{1-p}\, I,\ E_2 = \sqrt{p}X.$$

Phase flip channel.

$$\rho \to (1-p)\rho + pZ\rho\, Z^\dagger, \qquad E_1 = \sqrt{1-p}\, I,\ E_2 = \sqrt{p}Z.$$

Bit-Phase flip channel

$$\rho \to (1-p)\rho + pY\rho\, Y^\dagger, \qquad E_1 = \sqrt{1-p}\, I,\ E_2 = \sqrt{p}Y.$$

Depolarizing channel.

$$\rho \to (1-p)\rho + p\frac{I}{2}, \qquad E_0 = \sqrt{1-3p/4}\, I,\ E_1 = \sqrt{p/4}X,\ E_2 = \sqrt{p/4}Y,\ E_3 = \sqrt{p/4}Z.$$

For n qubits, the depolarizing channel is

$$\rho \to (1-p)\rho + p\frac{I}{2^n}.$$

Shor's code protects all 1-qubit errors! Now we use the formal description to describe how Shor's code correct all one-qubit errors that are of the form,

$$E_k = e_{k0}I + e_{k1}X + e_{k2}XZ + e_{k3}Z.$$

Let us quote a remark from Nielsen & Chuang:

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The apparent continuum of errors that may occur on a single qubit can all be corrected by
correcting only a discrete subset of those errors.
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The action of E_k on a qubit gives rise to superposition (of no error and three types of errors),

$$E_k|\psi\rangle = e_{k0}|\psi\rangle + e_{k1}X|\psi\rangle + e_{k2}XZ|\psi\rangle + e_{k3}Z|\psi\rangle.$$

We then consider formally using "syndrome" measurements for X and Z errors , and this act of measuring syndromes collapses to either of the four components (as the syndromes are classically distinct) and the corresponding correction operators can be applied to recover $|\psi\rangle$.

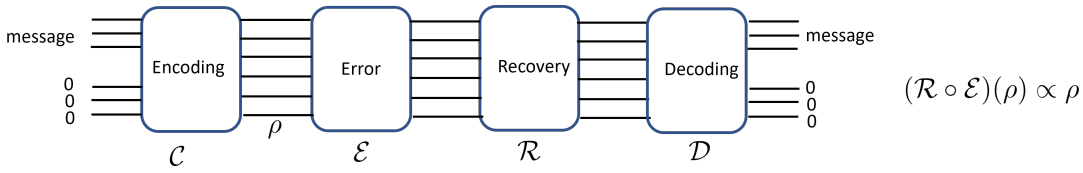


FIG. 3. Illustration of the advantage of error correction codes (i.e. the encoding).

IV. CORRECTABLE ERRORS

Let us consider the process of encoding, error occurrence $\mathcal{E}$, recovery (via syndrome measurement and further operation) $\mathcal{R}$ and the decoding; see Fig. 3. In order for the encoding be useful, the encoded state ρ, after the combination of $\mathcal{E}$ and $\mathcal{R}$, the resultant state must be at least proportional to the original one:

$$(\mathcal{R}\circ\mathcal{E})(\rho)\propto\rho.$$

We will denote by $\mathcal{C}$ the error correction code or the code space and by P the projector to the code subspace. We are interested in what error sets $\{E_k\}$ can be corrected. This can be answered by the following "correctable condition" that the set and the code satisfy,

$$PE_i^\dagger E_j P=\alpha_{ij}P,$$

where α_{ij} is any arbitrary hermitian matrix.

What is the meaning of the above condition? One can simplify this condition by further diagonalizing α: $\alpha_{ij}=\sum_k d_k u_{ik}u_{jk}^*$ and in terms of the new basis,

$$F_k\equiv\sum_j u_{jk}^*E_j,$$

the condition becomes

$$PF_k^\dagger F_l P=d_k\delta_{kl}P.$$

This means that the distinct errors take to the code to orthogonal subspaces and hence they are correctable.

A illuminating exercise is to check this condition in the Shor's code.

- Exercise 10.10 (N&C): Explicitly verify the quantum error-correction conditions for the Shor code, for the error set containing I and the error operators X_j, Y_j, Z_j for j = 1 through 9.

V. STABILIZER GROUP AND ERROR CORRECTION CONDITION

We saw in the 3-qubit bit flip code, there are three stabilizer operators Z_1Z_2, Z_2Z_3 and Z_3Z_1. In addition, the identity I is a trivial stabilizer operator. They form a group $G=\{I,Z_1Z_2,Z_2Z_3,Z_3Z_1\}$, and for any $g\in G$, $g|\psi\rangle=|\psi\rangle$, where $|\psi\rangle=a|000\rangle+b|111\rangle$. This particular group is 'generated' by Z_1Z_2 and Z_2Z_3, i.e. $G=\langle Z_1Z_2,Z_2Z_3\rangle$. Any element in the group G can be obtained by some consequence of multiplying any element in the generator set. Note also that there are two indepedent generators and three qubits; thus we have one logical qubit, and the number $1=3-2$, i.e. the number of qubit minus the number independent generators of the corresponding stabilizer group.

It is useful to define a Pauli group for one qubit:

$$G_1=\{\pm1,\pm i\}\times\{I,X,Y,Z\}.$$

The Pauli group for two qubits is defined as

$$G_2=\{\pm1,\pm i\}\times\{I,X,Y,Z\}\otimes\{I,X,Y,Z\}.$$

The factor $\{\pm1,\pm i\}$ is added such that all the combinations indeed form a group.

Recall the meaning of a group (where is there a group 'multiplication') is defined by the following properties. (1) Closed: for $g\in G$ and $g'\in G$, we have $gg'\in G$. (2) Identity: there exists a special element e_G such that

$ge_G = e_G g = g$. (3) Inverse: for any $g \in G$, there exists an element g' such that $gg' = g'g = e_G$ and the element g' is called the inverse of g. (4) Associativity: for $g_1, g_2, g_3 \in G$, we have $(g_1 g_2)g_3 = g_1(g_2 g_3)$.

Stabilizer group of Shor's code. It is straightforward to see that the stabilizer group of Shor's code is generated by the following elements: (1) $X_1X_2X_3X_4X_5X_6$, (2) $X_4X_5X_6X_7X_8X_9$ (the two first two from bit flip), (3) Z_1Z_2, (4) Z_2Z_3, (5) Z_4Z_5, (6) Z_5Z_6, (7) Z_7Z_8, and (8) Z_8Z_9 (the latter 6 from the phase flip). Given that there are nine qubits and eight independent generators, we have one logical qubit.

General stabilizer group. Naively, a stabilizer element $g|\psi\rangle = |\psi\rangle$ splits the Hilbert space into two halves and hence reduces the dimension by two. k independent stabilizer elements or generators reduces the dimension from 2^n for n qubits to 2^{n-k}, which corresponds to $n-k$ logical qubits. This is stated in Nielsen and Chuang's book,

- Proposition 10.5 (Nielsen & Chuang): Let $S = \langle g_1 ..., g_{n-k} \rangle$ be generated by $n-k$ independent and commuting elements from G_n (the n-qubit Pauli group), and such that $-I$ is not in S. Then V_S the space stabilized by the group S is a 2^k-dimensional vector space (effectively k qubits).

Note that a stabilizer group is an Abelian group, i.e. any two elements commute $gg' = g'g$. V_S is a k-qubit code space $C(S)$ defined by the stabilizer group S. We can choose two sets of k operators (logical Z's and X's),

$$\{\bar{Z}_1, \dots \bar{Z}_k\},\ \{\bar{X}_1, \dots \bar{X}_k\},$$

such that $\{g_1, g_2, \dots, g_{n-k}, \bar{Z}_1, \dots \bar{Z}_k\}$ are independent and commuting, and

$$[g_j, \bar{X}_l] = 0,\ [\bar{Z}_i, \bar{X}_{j\neq i}] = 0,\ \bar{Z}_i\bar{X}_i = -\bar{X}_i\bar{Z}_i.$$

$\bar{X}$'s are thus logical X operators and $\bar{Z}$'s are logical Z operators. For the case of Shor's code, we can choose

$$\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9, \quad \bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9.$$

Error Correction Conditions for Stabilizer group S. With the understanding of the stabilizer formalism, we can express the error correction conditions in terms of this formalsim. We will not prove it but state the theorem from Nielsen and Chuang:

- Theorem 10.8 (Nielsen & Chuang): (Error-correction conditions for stabilizer codes). Let S be the stabilizer for a stabilizer code $C(S)$. Suppose $\{E_j\}$ is a set of operators in G_n such that $E_j^\dagger E_k$ is not in $N(S) - S$ for all j and k, where $N(S)$ is the normalizer of S. Then $\{E_j\}$ is a correctable set of errors for the code $C(S)$.

Note that $N(S)$, normalizer group of S, contains elements E of G_n that preserve S, i.e. for $g \in S$, $EgE^\dagger \in S$. [In this case, $N(S)$ is equal to the 'centralizer' $Z(S)$, the group that commutes with all elements in S.]

Checking Shor's code. For the stabilizer group of Shor's code, $N(S) - S$ [set of elements in $N(S)$ but not in S] contains operators of weight at least three: $X_1X_2X_3$, $X_4X_5X_6$, $X_7X_8X_9$, $Z_1Z_4Z_7$, $Z_2Z_5Z_8$, etc (including $\bar{X}$ and $\bar{Z}$). $E_j^\dagger E_k$ from single-qubit errors are not in this set! Therefore, single-qubit errors can be corrected.

Note that the above set $N(S) - S$ has elements whose weight is at least three, and this number $d = 3$ is called the code distance. So Shor's code uses $n = 9$ qubits to encode $k = 1$ logical qubit with a code distance $d = 3$. Usually, a code distance d can tolerate $\lfloor (d-1)/2 \rfloor$ bits of errors. Shor's code is an $[[n = 9, k = 1, d = 3]]$ quantum error correction code.

VI. OTHER CODES

The stabilizer formalism is very useful. One can define quantum error correction codes using a stabilizer group or its (independent) generators. We give a couple of other codes.

7-qubit Steane code $[[7, 1, 3]]$. The stabilizer generators are

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IIIXXXX
IXXIIXX
XIXIXIX
IIIZZZZ
IZZIIZZ
ZIZIZIZ
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and we can see that $n = 7$ and $k = 1$, as there are six independent generators. The logical operators (which commute with the generators) are

$$\bar{X} = X_1X_2X_3X_4X_5X_6X_7, \quad \bar{Z} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7.$$

With these one can convince onself that $d = 3$ by considering the minimum weight in the set $N(S) - S$. Similar to Shor's code, the nontrivial operators in the generators are either all X or all Z. These codes are referred to as Calderbank-Shor-Steane (CSS) codes.

One natural question is how to write down the logical basis states in terms of the 7-qubit components? One approach is to use the projector

$$\hat{P} = \prod_{g \in \text{generators}} \frac{I+g}{2} |\psi_0\rangle,$$

and apply $\hat{P}$ to some initial state, e.g. $|0\ldots0\rangle$: $\hat{P}|0\ldots0\rangle$. This will generate a state that is stabilized by the stabilizer group S. Further logical zero and one are obtained by an additional projection to $(1 \pm \bar{Z})/2$ to this 'ground state'. A general method was proposed by Cleve and Gotteman using efficient quantum circuits: [Phys. Rev. A 56, 76 (1997)].

We thus display the logical states below,

$$\begin{aligned} |0_L\rangle &= \frac{1}{\sqrt{8}}\Big(|000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &\quad +|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\Big), \\ |1_L\rangle &= X_1X_2X_3X_4X_5X_6X_7|0_L\rangle. \end{aligned}$$

Steane code and classical codes. We can have the so-called Parity check matrix H of a classical code C:

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

where the code C is defined by the generator matrix G,

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

such that $HG = 0$.

This is equivalent to $G^T H^T = 0$. So we have the dual code $C^\perp$ defined by the H^T, whose parity check matrix is G^T. One can check (*as an exercise, by examine their columns*) that all code strings generated by H^T are contained in those by G, i.e. $C^\perp \subset C$.

The CSS quantum error correction code is defined by two classical codes C_1 and C_2 with $C_2 \subset C_1$ [in our case: $C_1 = C,\ C_2(= C^\perp) \subset C_1$] via

$$|x + C_2\rangle \equiv \frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} |x + y\rangle,$$

where x is any code string in C_1. One can verify directly that the above Steane code can indeed be obtained in this way.

The 5-qubit code $[[5, 1, 3]]$. The stabilizer generators are

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XZZXI
IXZZX
XIXZZ
ZXIXZ
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The logical operators are

$$\bar{X} = X_1X_2X_3X_4X_5, \quad \bar{Z} = Z_1Z_2Z_3Z_4Z_5.$$

The smallest code that corrects one-qubit error!

VII. AMPLITUDE AND PHASE DAMPING MODELS

Amplitude damping. The amplitude damping is a model for the spontaneous emission that an atom in the excited state can jump down to the ground state,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \; E_1 \equiv \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

here 0 or up is the ground state and the 1 or down is the excited state; γ is the strength. A qubit described by its density matrix ρ will become

$$\rho \to E_0\,\rho\,E_0^\dagger + E_1\,\rho\,E_1^\dagger =: \mathcal{E}(\rho).$$

For one qubit,

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \to \begin{pmatrix} |\alpha|^2+\gamma|\beta|^2 & \sqrt{1-\gamma}\,\alpha\beta^* \\ \sqrt{1-\gamma}\,\alpha^*\beta & (1-\gamma)|\beta|^2 \end{pmatrix}$$

We can analyze the efficacy of the error correction model: if it corrects one-qubit (?Pauli?) error, then it should reduce the amplitude damping error rate from γ to γ^2. (But we don't do it here.) In this case, the minimum fidelity is a useful quantity (comparing if the encoded case has better fidelity than the un-encoded case if errors occur), where the fidelity is defined as

$$F = \langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle.$$

Phase damping. The amplitude damping is a model for phase decoherence,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \; E_1 \equiv \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix},$$

where γ is the strength. A qubit described by its density matrix ρ will become

$$\rho \to E_0\,\rho\,E_0^\dagger + E_1\,\rho\,E_1^\dagger =: \mathcal{E}(\rho).$$

For one qubit,

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \to \begin{pmatrix} |\alpha|^2 & \sqrt{1-\gamma}\,\alpha\beta^* \\ \sqrt{1-\gamma}\,\alpha^*\beta & |\beta|^2 \end{pmatrix}.$$

VIII. LOGICAL GATES

We want to perform quantum computation using logical qubits and we do not want to do encoding and then decoding to peform universal set of gates before encoding again. We want to directly perform logical operation in the encoded space. Moreover, we want the logical operations (e.g. logical Hadamard, logical T gates, and logical CNOT) to be 'transversal', i.e., composed of product of local physical gates: $U_{\text{logical}} = \bigotimes_i u_i$. Transversal gates are inherently fault-tolerant because they prevent physical errors on one qubit from propagating to other qubits within the same logical code block.

There is unfortunately a no-go theorem by Eastin and Knill (2009), which states that no single quantum error-correcting code capable of detecting local physical errors can implement a universal set of logical gates using exclusively transversal operations. This result relies on Lie group theory and error detection properties. First, the space of all logical unitary transformations on n logical qubits forms a continuous Lie group $SU(2^n)$, and thus to achieve quantum computational universality requires generating a dense subgroup of $SU(2^n)$. But Eastin and Knill show that for any quantum code capable of detecting single-qubit errors (distance $d \geq 2$), the group of logical operations achievable via transversal physical unitaries is strictly discrete and finite, thus failing to achieve a dense continuous set of unitaries.

There have been some workarounds proposed: (1) Magic state distillation and injection; (2) Code Switching: converting between two different code structures that feature complementary transversal gate sets (e.g., switching between the 7-qubit Steane code for transversal Clifford operations and the 15-qubit Reed-Muller code for a transversal T gate); (3) Gauge Fixing: In subsystem codes (such as 3D color codes), gauge degrees of freedom are measured or fixed to alternate the logical subspace, enabling non-Clifford gates without requiring state distillation; (4) Lattice Surgery

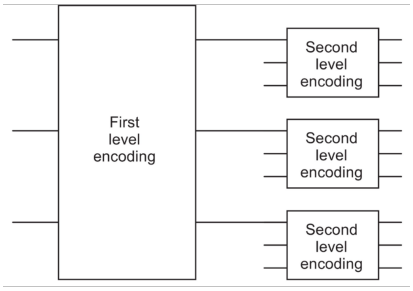


FIG. 4. Illustration of concatenation. [Source? N&C?]

& Defect Braiding: In topological surface codes, non-transversal logic is performed by modifying code boundaries, merging/splitting code blocks, or braiding topological defects.

Let us looks at some gates. Since we can implement logical Z gate $\overline{Z}$ and X gate $\overline{X}$, can we achieve logical Hadamard gate $\overline{H} = (\overline{X} + \overline{Z})/\sqrt{2}$? Can it be done with transversal gates? It depends on the error correction code. For example, the 7-qubit Steane code $\overline{X} = XXXXXXX$ and $\overline{Z} = ZZZZZZZ$, using $\overline{H} \equiv \prod_{i=1}^{7} H_i$, we can achieve $\overline{HXH} = \overline{Z}$. Moreover with $\bar{S} \equiv\equiv \prod_{i=1}^{7} S_i$, it gives $\overline{X} \to \prod_i Y_i := \overline{Y} \to (-1)^7 \prod_i X_i = -\overline{X}$, achieving a valid logical S gate.

How about CNOT? Suppose we have two Steane-code blocks (1 to 7 for the first and 8 to 14 for the second) and we perform CNOTs pairwise $\prod_{i=1}^{7} \mathrm{CNOT}_{i,i+7}$: then $\overline{X}_1 \to \overline{X}_1\overline{X}_2$ and $\overline{Z}_2 \to \overline{Z}_1\overline{Z}_2$. And it is a valid logical CNOT between the two logical blocks (or qubits). The Steane code admits the entire Clifford group as transversal gates.

How about T gate? It transforms a single qubit $X \to (X+Y)/\sqrt{2}$ and $\prod_i T$ conjugates $\overline{X}$ to $\prod_i (X+Y)_i/\sqrt{2}$, not a logical T. This is consistent with the Eastin-Knill theorem. An example code that has a transversal T gate is the $[[15, 1, 3]]$ quantum Read-Muller code (a CSS one with $d_X = 7, d_Z = 3$), which is constructed from two nested classical punctured Reed-Muller codes, $C_1 = RM(1,4)^*$ and $C_2 = RM(2,4)^*$ (Q: is it a product code?). In particular, it can be visualized on a tetrahedron, so that qubits are placed on all 4 vertices, mid-points of all 6 edges, centers of all 4 faces, and one point inside the tetrahedron: 4+6+4+1=15. The Z-stabilizers are 4-body on each quadralateral (12 of them, but only 10 are independent), and 4 indepdent X-stabilizers are on the volumes formed by the 8 vertices (each has vertices on 1 vertex of the tetrahedron, 3 triangular phases, 3 tetrahedron edges, and the center of the tetrahedron). T is not Clifford, so we need to check explicitly that it leaves the code space invariant. The logical $|\overline{0}\rangle$ consists of components with 8 one's and 7 zero's and the logical $|\overline{1}\rangle = X^{\otimes 15}|\overline{0}\rangle$, and $T^{\otimes 15}|\overline{0}\rangle = |0\rangle$ but $T^{\otimes 15}|\overline{1}\rangle = e^{-i\pi/4}|0\rangle$, which acts as a logical $T^\dagger$.

The 15-qubit Reed-Muller code provides a transversal non-Clifford gate:

$$\overline{T^\dagger} = T^{\otimes 15} = \bigotimes_{i=1}^{15} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Because all Z-type stabilizers have weights that are multiples of 8 (specifically, weight 8), applying the physical T gate across all 15 qubits commutes with the code space up to a logical $T^\dagger$ operation. Unfortunately, it does not have a transversal H gate.

IX. FAULT TOLERANCE AND ERROR THRESHOLD

If the error probability p of a gate is less than some threshold pth p_{th}, then arbitrarily long quantum computations are possible using noisy gates, with a reasonable overhead cost. This uses 'concatenation', i.e. many layers of the same encoding; see Fig. 4. Suppose that each layer encoding reduces the error probability p to cp^2. Then for a total of k layers, the reduction is

$$p \to p^{[k]} \equiv \frac{1}{c}(cp)^{2^k}.$$

We want that the error probability to be smaller than the gate error rate per polynomial number of gates, i.e. $p(k) \le \epsilon/\mathrm{poly}(n)$. This requires at least that $p^{[k]} < p^{[k-1]}$, and thus

$$p < p_{\mathrm{th}} = 1/c.$$

The exact value of p_{th} and the overhead depend on the error model and the fault-tolerant scheme.

X. TOPOLOGICAL CODES

The idea of topological codes is to use codes that do not require active error correction but just passive error protection. By now there are many codes, including the Toric code, color codes, Bacon-Shor codes, subsystem codes, fracton codes, Levin-Wen string-net models, etc. They are closely related to intrinsic topological phases of matter, where one key feature is the emergence of exotic excitations called anyons.

Kitaev's toric code. This is one of the earliest topological codes and originally constructed on a torus and later extended to planar surfaces, with the latter called the surface code. However, these codes do not give universal gates and additional work needs to be done in order to make quantum computation on these universal, such as the magic state distillation, which we discuss in a later unit. We will probably postpone the detailed discussions on Kitaev's toric code to the next unit.

A. From Shor's code to surface code and to Bacon-Shor code

We can arrange Shor's code on horizontal edges a 3x4 grid. It can be generalized to $n \times (n+1)$ grid. There are long X stabilizers, e.g., on two neighboring rows of edges. To reduce them to local stabilizers, we insert X on the vertical edges between the two rows. Doing this for all neighboring rows, X stabilizers are broken down to four X's (around a plaquette) in the bulk or three X's on the boundaries, and Z stabilizers expand for two Z's to four Z around a vertex. This is a surface code $[[n^2 + n(n-1), 1, n]]$. If we make the boundary periodic on both directions, we obtain a toric code, which has two qubits. In a genius g surface, we have $|E| - (|V| - 1) - (|F| - 1) = 2g$ logical qubits. It is a CSS code and we have two parity check matrices that satisfy $H_Z H_X^T = 0 \pmod 2$. This defines a chain complex. More detail on the toric code in the next unit.

Bacon-Shor:

XI. ERROR MITIGATION (NOT ACTIVE CORRECTION)

Gate error mitigation. This was originally proposed by two groups in two separate works; see Refs. [1, 2]. A gate is achieved by some evolution operator via external field or coupling strength $J(t)$. Ideally, the same area will correspond to the same ideal gate; see Fig. 5. In reality, longer pulses incur more decoherence; we shall use λ to denote the noise strength. For an observable A, we can expand its expectation value in terms of λ,

$$\mathrm{Tr}(A\rho(t=T)) = E_K(\lambda) = E^* + \sum_{k=1}^{n} a_k \lambda^k + R_{n+1}(\lambda, \mathcal{L}, \mathcal{T}).$$

Then we can use different pulses to mimic different noise strengths by varying their lengths: $c_j\lambda$; from several points of c_j, we can extract ideal E^* up to small correction,

$$E^* \approx \sum_{j=0}^{n} \gamma_j E_K(c_j\lambda) + \mathcal{O}(\lambda^{n+1}),$$

where the coefficients γ_j's are chosen such that

$$\sum_j \gamma_j = 1, \ \sum_{j=0}^{n} \gamma_j c_j^k = 0.$$

This method is called Richardson extrapolation.

One much simplified approach that relies on the assumption that single-qubit gates are of high fidelity and the most errors come from the CNOT gate. Then one replace any CNOT gate in the original circuit by $q = (2k-1)$ CNOTs and measure the expectation values of the desired observables. Then one extrapolates to the limit $q \to 0$ to obtain the ideal observable values for the noiseless CNOT limit [3].

Measurement error mitigation. Basic idea: prepare computational states n and measure in computational basis. Gather enough statistics matrix M; see, e.g., Ref. [4]; also a standard procedure in Qiskit.

$$\tilde{P}_{(n_0,n_1,\ldots,n_{N-1})}[\text{measured}] \ = \ \sum_{\vec{m}} \mathbf{M}_{\vec{n};\vec{m}} \, P_{\vec{m}}[\text{ideal}]$$

One only needs to invert the eqution to find P[ideal] with the constraint that it is non-negative.

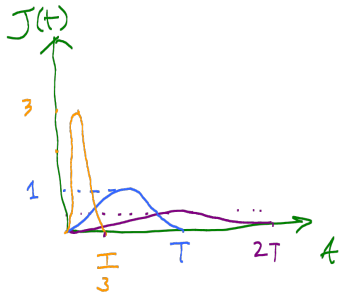


FIG. 5. Illustration of the pulse $J(t)$ and its integrated area.

XII. CONCLUDING REMARKS

In quantum computers, there are more errors than just bit flip and phase. But due to quantum superposition, being able to correct flip and phase errors allows to correct all one-qubit errors, which is an amazing result of quantum error correction. Quantum computers will spend more effort in preventing and actively correcting errors than classical ones; we need measurement to find errors and apply correcting operations. Quantum error correction has been well developed, some of which drawing inspiration from classical coding theory. Now QECC is also used in many other fields, e.g. condensed matter physics and AdS/CFT holographic entanglement.

In this unit, we have discussed error models, quantum error correction, topological stabilizer codes and topological phases, and error mitigations.

It is a good time to check whether you have achieved the following Learning Outcomes:
After this Unit, You'll be able to understand why quantum information is fragile but quantum correction codes can be used to reduce error rates in logical qubits.

Suggested reading: N&C chap 10; KLM chapter 10; Qb chap 5.1-5.2

Appendix A: To be added

Eastin-Knill theorem: a no-go theorem for the transveral universal gate set. Transveral gates are important because they ensure that errors will not spread too much so that error correction is possible. In particular, each physical qubit in a code block is acted on by at most a single gate. The theorem states that a universal set such as {H, S, CNOT, T } cannot be implemented transversally. Given this, we ask how fault-tolerant quantum computation can still be possible?

One another important subject that should have been discussed is the qRAM, which should have appeared in the Grover's algorithm on unstructured data search. How to implement qRAM in a fault-tolerant, efficient, and practical way is still an important research topic.

Appendix B: Code switching between Steane and Read-Muller

Code switching between the $[[7, 1, 3]]$ Steane code (7 physical qubits) and the $[[15, 1, 3]]$ Reed-Muller code (15 physical qubits) solves the qubit count mismatch through subsystem gauge fixing. Both codes can be viewed as different "gauges" (fixing of unencoded degrees of freedom) of an underlying 15-qubit subsystem code containing 1 logical data qubit and 3 gauge qubits. The 8-qubit difference is handled by dynamically introducing or discarding 8 ancillary physical qubits.

1. Underlying Gauge StructureSteane Code Gauge: Fixing the 3 gauge qubits in the Z-basis disentangles 8 physical qubits into product states. These 8 qubits can be removed, leaving the logical data encoded in the 7 physical qubits of the Steane code.Reed-Muller Code Gauge: Fixing the 3 gauge qubits in the X-basis couples all 15 physical qubits into the stabilizer space of the $[[15, 1, 3]]$ Reed-Muller code.

2. Step-by-Step Conversion Protocol

```
[[7,1,3]] Steane (7 Qubits): 1. Append 8 Ancilla Qubits (in 0)  --> Total 15 Qubits;
2. Measure X-type Gauge Generators; 3. Apply Pauli Corrections. Leads to
```

```
[[15,1,3]] Reed-Muller Code (15 Qubits)  <-- Apply Transversal T Gate:
1. Measure Z-type Gauge Generators;
2. Apply Pauli Corrections;3. Discard 8 Disentangled Qubits. Back to
[[7,1,3]] Steane Code (7 Qubits)
```

Up-Conversion: 1. Steane → Reed-Muller (7 → 15 Qubits); 2. Ancilla Preparation: Append 8 ancillary physical qubits initialized to a known state (e.g., $|0\rangle^{\otimes 8}$) to the 7-qubit Steane code block, bringing the total count to 15 qubits; 3. Gauge Measurement: Fault-tolerantly measure the 3 X-type gauge generators (or the extra X-type stabilizers) unique to the 15-qubit Reed-Muller code; 4. Correction: Apply Pauli Z corrections based on the measured syndrome outcomes; 5. Result: The system is projected into the $[[15, 1, 3]]$ code space. The transversal non-Clifford gate ($\overline{T} = T^{\otimes 15}$) can now be executed.

Down-Conversion: 1. Reed-Muller → Steane (15 → 7 Qubits); 2. Gauge Measurement: Fault-tolerantly measure the Z-type gauge generators (or perform single-qubit Z measurements on the 8 auxiliary qubits); 3. Correction: Apply Pauli X corrections based on the outcomes; 4. Decoupling: The Z-basis measurement disentangles the 8 auxiliary qubits from the logical data; 5. Result: The 8 auxiliary qubits are discarded, returning the logical state to the $[[7, 1, 3]]$ Steane code on the remaining 7 physical qubits, ready for transversal Clifford operations ($H, S, CNOT$).

Maintaining Fault Tolerance: Throughout the conversion, the code distance $d = 3$ is preserved. Because the gauge operators and stabilizers are measured using fault-tolerant extraction gadgets (such as Shor or flag-qubit circuits), any single physical error ($t = 1$) occurring during the process is detected and corrected by the syndrome measurements without corrupting the logical qubit.

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