 Electron, Rutherford’s nuclear atom, and quantization of light

We are all familiar with atoms and molecules, which are building blocks of everyday materials. Gold (Au), Silver (Ag), Copper (Cu), Aluminum (Al), Tin (Sn), etc. are metals. Air is composed of nitrogen molecules (N₂) and oxygen molecules. You probably still remember the ideal gas law \[ PV = nRT = N k_B T \]. A water molecule is composed of two H and one O: H₂O is a macroscopic form of H₂O that can exist in the form of ice (solid), water (fluid) and vapor (gas).

The Periodic Table lists all elements.

\[ ^{15}_{15} n = 1 \text{fm} \]

1 mole = 6.022 x 10²³ objects

\[ \text{1 u} = \frac{1}{12} \text{ (mass of } ^{12} \text{C atom)} \]

\[ Z = \text{atomic number} \]

\[ A = \text{mass number} \]

\[ N = \text{neutron number} \]

\[ Z = n + N \]

\[ B = (Z m_e + Z m_p + N m_n - M_{\text{atom}})^2 \]

We will discuss 3 topics from chapter 3 (Atoms):

3.10: Thomson’s Discovery of electron
3.11: Millikan’s Oil-Drop experiment
3.12: Rutherford & the nuclear atom
Thomson's Discovery of \( e^- \)

1. When a\( V \) voltage is applied, some gas atoms ionize and electrons disperse, \( e^- \) accelerates and moves toward the anode and passes through. 

2. \( \vec{F} \) with deflecting plates, \( e^- \)'s are bent.

The ray can be used to change a metal cup.

Also bent by a magnetic field, \( \vec{B} \).

(i) Using combinations of \( E \) & \( B \), one can measure \( v \) of \( e^- \):

\[
\vec{F} = -e (\vec{E} + \vec{v} \times \vec{B})
\]

\[
\vec{F} = \frac{e}{m} \vec{v} \times \vec{B}
\]

(ii) With \( v \) known, can measure radius of circular motion in field \( B \):

\[
R = \frac{m v}{eB} \quad \text{known}
\]

\[
\Rightarrow \text{infer } \frac{v}{eB}
\]

3.40 In Thomson's experiment electrons travel with velocity \( v \) in the \( x \) direction. They enter a uniform electric field \( E \), which points in the \( y \) direction and has total width \( l \) (Fig. 3.15). Find the time for an electron to cross the field and the \( y \) component of its velocity when it leaves the field. Hence show that its velocity is deflected through an angle \( \theta \approx eEI/(mv^2) \) (provided that \( \theta \) is small). Assume that the electrons are nonrelativistic, as was the case for Thomson.
3.41 Suppose that the electrons in Thomson’s experiment enter a uniform magnetic field \( B \), which is in the \( z \) direction (with axes defined as in Fig. 3.15) and has total width \( l \). Show that they are deflected through an angle \( \theta \approx eBl/(mv) \) (provided that \( \theta \) is small). Assume that the electrons are nonrelativistic.

\[
\frac{mv^2}{R} = eVB, \quad R = \frac{mv}{eB}
\]

\[
\sin \theta = \frac{l}{R}
\]

\[
\theta \approx \frac{leB}{mv}
\]

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**Millikan’s Oil-drop Experiment**

Thomson’s experiment can only determine the ratio \( m/e \), not \( m \) or \( e \) individually. Millikan was able to measure \( e \).

Oil drops will fall under gravity but due to friction viscosity will reach a terminal velocity \( V = \frac{Mg}{6\pi \eta R} = \frac{4\pi \eta p g}{6\pi \eta R} = \frac{2\pi \eta p g}{9 \eta} \). Known \( M \) inferred from measuring \( R \)

But with \( e^- \) and external field \( E \) to balance the force (adjust it)

\( pE = Mg \Rightarrow \rho = \frac{4\pi \eta p g}{E} \)

\( e = 1.6 \times 10^{-19} \) coulomb

Electron’s charge \( e = -e \)

\( m_e = 9.11 \times 10^{-31} \text{ kg} = 0.511 \text{ MeV}/c^2 \)

\( \sim \frac{1}{2000} \) of hydrogen atom
**Rutherford's experiment and the nuclear atom**

Thomson had a model that $e^-$ is distributed in uniform positive background (of sphere of atom).

Rutherford and his assistant Geiger & Marsden carried out experiments to test this.

![Diagram of Rutherford scattering experiment](image)

**FIGURE 3.8**
The "Rutherford scattering" experiment of Geiger and Marsden. Alpha particles from a radioactive source pass through a narrow opening in a thick metal shield and impinge on the thin foil. The number scattered through the angle $\theta$ is counted by observing the scintillations they cause on the zinc sulfide screen.

**FIGURE 3.11**
Geiger and Marsden measured the flux of scattered alpha particles at 14 different angles. Their measurements fit Rutherford's predicted $1/\sin^4(\theta/2)$ behavior beautifully. Notice the vertical scale is logarithmic and the measurements span an amazing 5 orders of magnitude.

- Used $^{9}\text{Be}^+$ particle to bomband on thin metal foil.
- Most deflected angles $\theta$ are small, but some are as large as $90^\circ$ and even $180^\circ$ (even through foil).
- Not consistent with uniform positive charge.

Rutherford concluded with a model of atom:

- A nucleus carrying positive charge in a small volume and $e^-$ outside the positive nucleus.

Can this lead to some quantitative prediction to compare with experiment?
Rutherford derived a formula for \# of particles per unit area at \( \theta \)
\[ n_{se}(\theta) = \frac{N \int_{0}^{t} \left( \frac{Zke^2}{E} \right)^2 \cdot \frac{1}{\sin^4(\theta/2)} \cdot 4s^2 \cdot \text{d}t}{4s^2} \]  
(Rutherford formula)

\[ b = \frac{2ke^2}{E \cdot \tan \theta} \]
\[ b: \text{impact parameter} \]

Derived using:
- Coulomb repulsion \( F = \frac{kQq}{r^2} = 2Zke^2 \)
- Target foil has thickness \( t \) and contains \( N \) atoms in unit volume.
- \( s \): distance from nucleus to detector
- \( E \): energy of \( \alpha \) particle

[For derivation, see 3.12.] The formula agrees w. data!

But with \( \theta \)'s \( \otimes \) orbiting nucleus is not stable acc. to E&M
Quantization of light

According to classical theory of radiation, the energy of light is continuous. However, experiments around 1900 indicated failure of such theory to explain the so-called black-body radiation, which was one of many catastrophes of classical mechanics!

Max Planck was able to explain the observed distribution by invoking the notion that light with frequency $f$ has energy in chunks:

$E = n \hbar f$, $2n \hbar f$, $3n \hbar f$

(with $\hbar$ called Planck constant)

$\hbar = 6.63 \times 10^{-34}$ J·s

$I(\lambda, T) = \frac{2\pi \hbar c^2}{\lambda^5} \left( \frac{1}{e^{\frac{\hbar c}{\lambda k T}} - 1} \right)$

power radiated per unit area with wave length $\lambda \leq \lambda_0$
Examples of black-body radiation include our sun and the so-called cosmic microwave background (CMB); the latter provides the evidence that our universe experienced a period of fast expansion. (inflation?)

[We will be able to derive this when we learn quantum statistical mechanics.]

\[ \Rightarrow \text{Energy of light is quantized} \Rightarrow \text{notion of photons} \]

**Photoelectric effect**

Einstein took up Planck's idea and used it to explained the photoelectric effect. (by Heinrich Hertz in 1887)

![Diagram of photoelectric effect]

- Metal exposed to light can eject electrons
  \[ \Rightarrow \text{a of } e^- \]
  (detected by a current if properly biasing 2nd electrode)
- 2nd electrode kept at lower potential \( \Rightarrow \) repel \( e^- \)
  at a stopping potential \( V_s \) : current ceases

\[ V_s e = K_{\text{max}} \] (the potential energy balances the max kinetic energy)

\[ \& 1. \text{ If intensity of light is increased, # of } e^- \text{ increases, but} \]
  their kinetic energy does not change!

\[ \& 2. \text{ If frequency } f \text{ is below a threshold } f_0 \Rightarrow \text{no } e^- \text{ ejected no matter how high intensity} \]
These cannot be explained by a continuous energy of light (i.e. proportional to intensity). But quantization of light energy $E = hf$ can easily explain $1 \text{ a. z.}$

$$h \nu_0 = \phi \Leftarrow \text{ work function, i.e. minimum energy needed to overcome potential energy of } e^- \text{ inside metal.}$$

$$hf - \phi = K_{\text{max}}$$

**Example 4.1: Typical visible photon? How many photons in light intensity?**

\[ \text{E.g. } \lambda = 700 \text{ nm} \]

\[ \lambda \approx 550 \text{ nm}, \quad E = hf = \frac{hc}{\lambda} \]

\[ h = 6.63 \times 10^{-34} \text{ J.s}, \quad 3 \times 10^8 \text{ m/s} \]

\[ E = \frac{1240 \text{ eV}}{550} \approx 2.3 \text{ eV}. \]

\[ E = I \cdot A = 3 \times 10^{-4} \text{ W/m}^2 \cdot 3 \times 10^{-5} \text{ m}^2 \approx 9 \times 10^{-9} \text{ W} \]

\[ \text{E.g.: 6 mm diameter, } \text{H pen second} \approx \frac{9 \times 10^{-9} \text{ J}}{1.989 \times 10^{-15} \text{ J}} \approx 2.5 \times 10^9 \]

**Example 4.2: \( \phi = 4.7 \text{ eV} \) silver \( V_s = ? \) for \( \lambda = 200 \text{ nm} \)**

\[ V_s = \frac{hf - \phi}{e} = \frac{hc}{\lambda e} - \frac{\phi}{e} = \frac{1240 \text{ eVnm}}{200 \text{ nm}} - 4.7 \text{ eV} = \left(6.2 - 4.7\right) \text{ eV} = 1.5 \text{ eV} \]

### 4.4 X-rays and Bragg Diffraction

X-rays are light with wavelength

\[ 0.01 \text{ nm} \sim 1 \text{ nm} \rightarrow \text{ high energy} \]

\[ 1240 \text{ eV} = 1240000 \text{ eV}! \]

Generated by high speed electrons colliding onto metal \( \{ \text{ Bremsstrahlung = braking radiation} \} \)
Due to small wavelength (can be comparable to spacing between atoms in crystal), X-rays are good for probing crystal structure, using interference, specifically called Bragg diffraction.

\[ 2d \sin \theta = n \lambda \] (Bragg law)

**FIGURE 4.7**
An X-ray spectrometer. X-rays are reflected off the crystal and detected by the ionization chamber. The crystal and chamber can both rotate in such a way that the two angles \( \theta \) are always equal.

Crystal structure, DNA structure...
X-ray spectra

\[ hf_{\text{max}} = k = V_0 e \]

The Duane-Hunt law

### 4.5 X-ray Spectra

![Graph showing X-ray spectra produced by platinum and molybdenum anodes, both made with an accelerating potential of 35 kV. Note how both spectra terminate abruptly at the same frequency, \( f_{\text{max}} \).]

**FIGURE 4.11**

Classical theory of bremsstrahlung predicts a broad spread of frequencies with smooth intensity.

But observed: \( \bullet \) Sharp peaks

\( \circ \) Exists a threshold \( f_{\text{max}} \)

\[ hf_{\text{max}} = K = V_0 e \]

**FIGURE 5.7**

Moseley measured the frequencies \( f \) of the \( K_{\alpha} \) X-rays, using several different elements for the anode of his X-ray tube. The graph shows clearly that \( \sqrt{f} \) is a linear function of the atomic number \( Z \) of the anode material. The reason the line crosses the \( Z \) axis at \( Z \approx 1 \) is explained in the text.

\[ \sqrt{f} \propto (Z - 1) \]

\[ E_\gamma = \frac{3}{4} (Z - \delta)^2 E_R \]

Transitions between \( n = 2 \) and \( n = 1 \) are traditionally identified as the \( K_{\alpha} \) frequencies of their \( K_{\alpha} \) X-rays (or any other definite X-ray line), then the photon energies, and hence frequencies, should vary like the square of the atomic number \( Z \); that is, we should find \( f \propto Z^2 \), or equivalently

\[ \sqrt{f} \propto Z \]

(5.35)
Classical theory predicts that when a light source is fired at a system of charged particles, the frequency of the scattered light is $f = f_0$ (classical).

But this is not what was observed. A range of frequencies (or equivalent wave lengths) was seen. (Careful measurement)

Moreover, it was also seen that $f < f_0$.

Compton was able to explain this using quantization of light. For light ($m = 0$),

$$E = (pc)^2 \pm (mc^2)^2 = (pc)^2$$

Solve for $p$

$$p = \frac{E}{c} \Rightarrow p = \frac{R_f}{c} \Rightarrow p = \frac{R_f}{\lambda}$$

Assume photon collides with $e^{-}$ just like two particle collision, momentum and energy are conserved.

Momentum:

$$p_o = \frac{E_0}{\lambda_0} = p_c \cos \theta + p_e \sin \theta$$

$$0 = p_{s-o} - p_e \sin \theta$$

(or $p_o = p_e + p$)

Energy:

$$p_o c + mc^2 = p_c + \sqrt{p_e^2 + mc^2}$$

From momentum:

$$p_e = (p_0 - P) (P_0 - P)^2 = p_o + p^2 - 2p_0 p - 2p_0 P \cos \theta$$

or

$$p_e \cos \theta = p_o - P \cos \theta \Rightarrow p_e = (p_0 - P \cos \theta)^2 + (p \sin \theta)^2$$

or

$$p_e \sin \theta = p \sin \theta$$
From energy \[ \sqrt{p_e^2 c^2 + m_e^2 c^4} = \sqrt{(p_0 - p)^2 c^2 + 2(p_0 - p) m_e c^2 + m_e^2 c^4} \]
\[ p_e^2 c^2 + m_e^2 c^4 = (p_0 - p)^2 c^2 + 2(p_0 - p) m_e c^2 + m_e^2 c^4 \]
\[ p_e^2 + m_e^2 c^2 = (p_0 - p)^2 + 2(p_0 - p) m_e c + m_e^2 c^2 \]
\[ p_e^2 + m_e^2 c^2 = (p_0 - p)^2 + 2(p_0 - p) m_e c + m_e^2 c^2 \]
\[ p_e^2 + m_e^2 c^2 = \left( \frac{p_0 - p}{m_e c} \right)^2 + \frac{2(p_0 - p) m_e c}{m_e^2 c^2} + \frac{m_e^2 c^2}{m_e^2 c^4} \]
\[ \Rightarrow m_e c (p_0 - p) = p_0 P (1 - \cos \theta) \]
\[ \Rightarrow \frac{1}{P} - \frac{1}{P_0} = \frac{1}{m_e c} (1 - \cos \theta) \]
\[ \Rightarrow \frac{\lambda}{P} - \frac{\lambda_0}{P_0} = \frac{1}{m_e c} (1 - \cos \theta) \]
\[ \Rightarrow \Delta \lambda = \lambda - \lambda_0 = \frac{\hbar}{m_e c} (1 - \cos \theta) \geq 0 \]
\[ \text{Since } \lambda \geq \lambda_0 \Rightarrow f \leq f_0 ! \]

Increase, \( \Delta \lambda \), in the wavelength of photons in Compton scattering. Note that \( \Delta \lambda \) is zero at \( \theta = 0 \) and rises to a maximum of \( 2 \hbar/m_e c \) at \( \theta = 180^\circ \).

**Figure 4.13**

Wave-particle duality

Light, usually regarded as a wave, displays particle-like properties, such as in Compton effect & photoelectric effect.

Light properties \( \lambda \) \& \( f \) \( \Rightarrow \) particle-like \( p = \frac{\hbar}{\lambda} \) \& \( E = h f \)

- In fact de Broglie reversed this:
  - particles also exhibit wave properties (6.2 De Broglie's hypothesis)

\[ p, E \Rightarrow \lambda = \frac{E}{p}, \quad f = \frac{E}{h} \]

**Figure 6.3**

Diffraction rings produced by diffraction of waves in polycrystalline metal samples with (a) X-rays, (b) electrons, (c) neutrons.