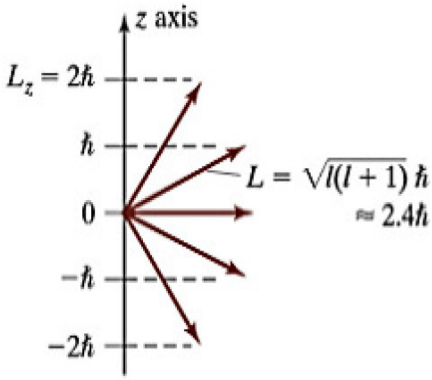
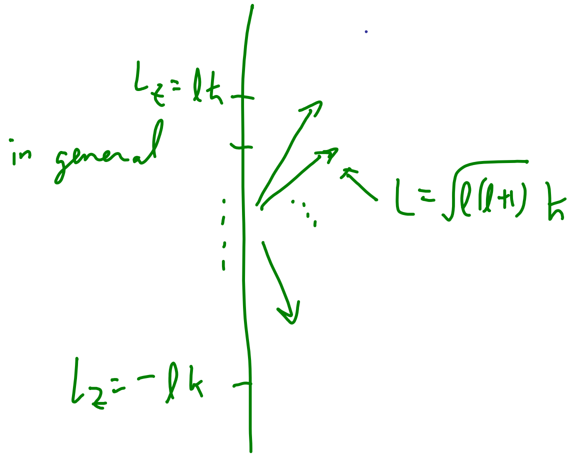
Electron spin

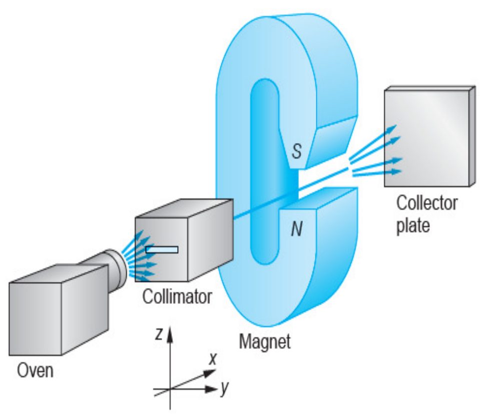
Vector model : we have seen the electron's orbital angular momentum

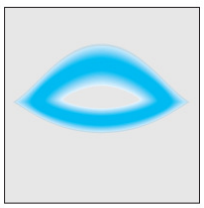




It has another quantity called spin, which is a property that is the intrinsic angular momentum, usually labeled by S.

The electron spin was discovered in an experiment by Stern and Gerlach





for hydrogen and silver atoms

$\Rightarrow$ two distinct values

We need to understand electron's spin in order to understand multielectron atoms in the periodic table. [The nucleus also has its spin and spin is important in Magnetic Resonance Imaging.]

Electron's spin & orbital angular momenta.

$$\underset{\text{total}}{\vec{J}} = \underset{\text{(orbital)}}{\vec{L}} + \underset{\text{(spin)}}{\vec{S}}$$

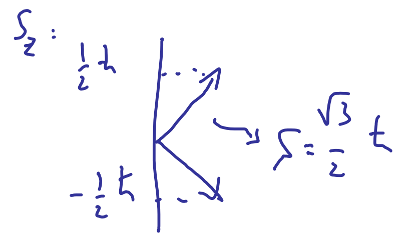
$\vec{L} \longleftrightarrow \vec{r} \times \vec{p}$ $\qquad$ $\vec{S} \longleftrightarrow I\vec{\omega}$

We learned that $\vec{L}$ has magnitude $L = \sqrt{l(l+1)}\,\hbar$ and

$L_z = m\hbar$ where $m = -l, -l+1, \dots\, l$ (integers)

Spin $\vec{S}$ has magnitude $S = \sqrt{s(s+1)}\,\hbar$ and $S_z = m_s \hbar$

where $m_s = -s, -s+1, \dots, s$



For electron spin $s = \frac{1}{2}$! $m_s = \pm\frac{1}{2}$!

$S = \frac{\sqrt{3}}{2}\hbar$, $S_z = \pm\frac{1}{2}\hbar$ $\uparrow / \downarrow$

$\Rightarrow$ important consequence : e^- state in hydrogen atom has extra label:

$\psi_{n,l,m,\underset{\uparrow/\downarrow}{m_s}}$ $\Rightarrow$ total degeneracy $= 2\cdot n^2$

factor of 2 from $\uparrow$ & $\downarrow$

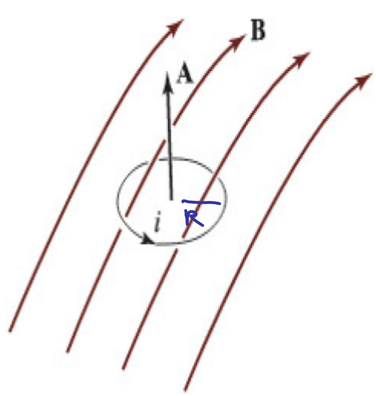
e.g. proton & neutron have spin $\frac{1}{2}$ as well.

$s = \frac{1}{2}$ $S = \frac{\sqrt{3}}{2}\hbar$

photon is spin 1. Δ has spin $\frac{3}{2}$ $(s = \frac{3}{2})$ $S = \sqrt{s(s+1)}\,\hbar$

Magnetic moment

A current loop carries a magnetic moment



$\mu = i\pi R^2 = iA$ (direction right thumb)

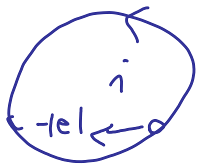
Such a loop in B field will experience

a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$.

work: $W = -\int \tau\, d\theta = -\mu B \int \sin\theta\, d\theta = \mu B \cos\theta + \text{const}$

$U(\theta) = -\mu B \cos\theta = -\vec{\mu}\cdot\vec{B}$

Since the current in the loop is carried by e^-'s and they go around the loop ⇒ have angular momentum



Consider one e^-:

$$i = \frac{e}{T} = \frac{v e}{2\pi R} = \frac{m_e v R}{2\pi R^2}\frac{e}{m_e} = \frac{e/m_e}{2A} L$$

$$\vec{\mu} = i\vec{A} = -\frac{e}{2m_e}\vec{L} \quad \text{opposite direction}$$

$$\left|\frac{\mu}{L}\right| = \frac{e}{2m_e} \quad \text{gyromagnetic ratio}$$

$$\vec{\mu} = -\frac{e}{2m_e}\vec{L}$$ holds for quantum case (even though consideration is really classical)

Zeeman effect (1896)

Using the potential energy $U = -\vec{\mu}_{orb}\cdot\vec{B} = +\frac{e}{2m_e}\vec{L}\cdot\vec{B}$

This means that the energy level in the absence of $\vec{B}$ E_0 will be spilt when $\vec{B}$ is introduced $E = E_0 + \Delta E$, with

$$\Delta E = \frac{e}{2m_e}\vec{L}\cdot\vec{B} \underset{\vec{B} = B\hat{z}}{=} \frac{eB}{2m_e}\underbrace{L_z}_{m\hbar} \qquad m = -l, -l+1, \dots, l$$

$m\hbar$: $2l+1$ different values

$$= m\frac{e\hbar}{2m_e}B = m\mu_B B$$

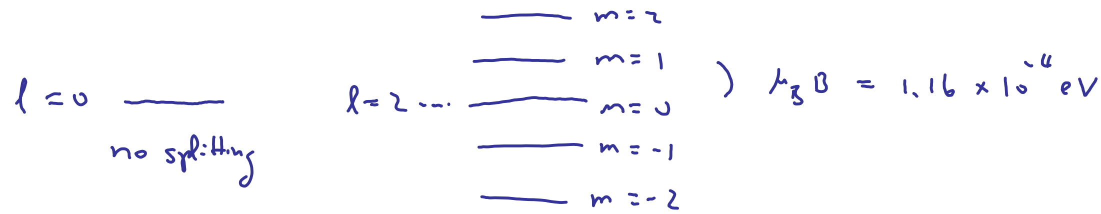
$$\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \mathrm{A\cdot m^2}$$ Bohr magneton.

$$= 5.79 \times 10^{-5}\ \mathrm{eV/T}$$

Example 9.1

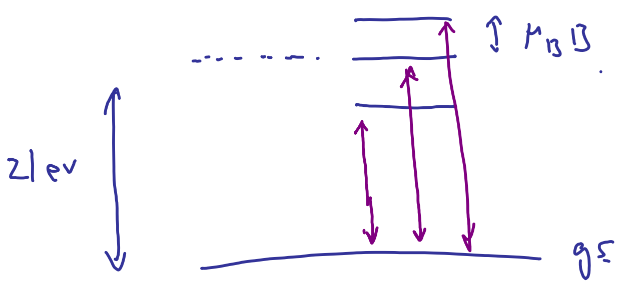
A helium atom is in one of its singlet states (with the two spins antiparallel and hence no spin magnetic moment). One of its electrons is in an s state ($l = 0$) and the other a d state ($l = 2$). The atom is placed in a magnetic field, $B = 2$ T (by normal laboratory standards a fairly strong field). By how much does the magnetic field change the atom's energy?

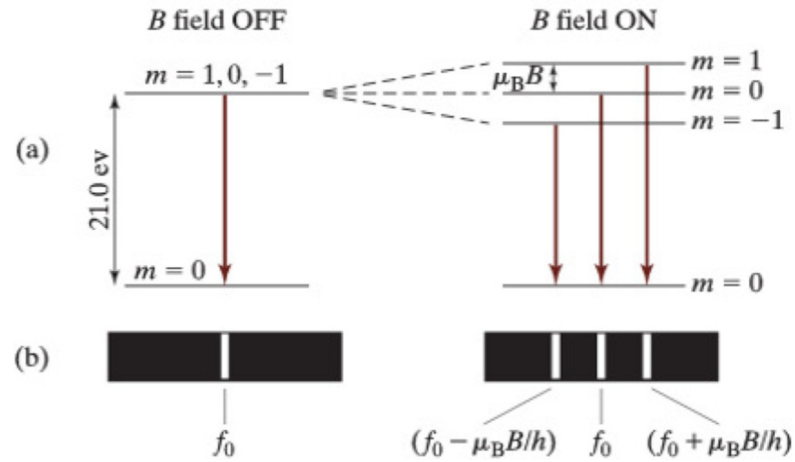
$\Delta E =$



Another example : ^4He $2e^-$: ① ground state both e^- in $l = 0$

② one low lying excited state $l = 1$ (one $l = 0$ & other $l = 1$)





Example 9.2

What is the wavelength λ_0 of the transition shown on the left of Fig. 9.4? If a magnetic field of 2 T is applied to the helium atom, what are the shifts $\Delta\lambda$ of the outer two spectral lines on the right of Fig. 9.4(b)?

$h\frac{c}{\lambda_0} = E_\gamma \qquad \lambda_0 = \frac{hc}{E_\gamma} = \frac{1240 \text{ eV}\cdot\text{nm}}{21 \text{ eV}} \approx 59 \text{ nm}$

$$\begin{array}{r} 59.04 \\ 21 \overline{)1240} \\ 105 \\ \hline 190 \\ 189 \\ \hline 10 \end{array}$$

$\lambda_\pm = \frac{hc}{E_\gamma \pm \mu_B B} \qquad \Delta\lambda = \lambda_\pm - \lambda_0 = \frac{hc}{E_\gamma}\left(1 \mp \frac{\mu_B B}{E_\gamma}\right) - \frac{hc}{E_\gamma}$

$$\begin{array}{r} 0.0055 \\ 21 \overline{)1.16} \\ 105 \\ \hline 110 \end{array}$$

$|\Delta\lambda| = \frac{hc}{E_\gamma} \cdot \frac{\mu_B B}{E_\gamma} \approx 59 \text{ nm} \cdot \frac{1.16 \times 10^{-4} \text{ eV}}{21 \text{ eV}} = 59 \text{ nm} \cdot 5.5 \times 10^{-6}$

$\approx 3.3 \times 10^{-4}$ nm $\Rightarrow$ small (not easy to detect around 1900)

Today, spectrometers can resolve splittings of order 10^{-8} nm, and the Zeeman shifts can be measured very accurately. An important modern application is to measure the splitting of an identified spectral line and hence to find an unknown magnetic field. This is especially useful in astronomy since the magnetic fields of the sun and stars cannot be measured directly.

Spin magnetic moment

$\vec{\mu}_{orb} = -\frac{e\hbar}{2m_e}\vec{L}$ natural extention: $\vec{\mu}_{spin} = -\gamma\vec{S}$

γ: spin gyromagnetic ratio

experimentally it was found

$\gamma = \frac{e\hbar}{m_e}$ i.e. twice that of orbital part (2.0023 times to be more precise)

total magnetic moment for an e^-:

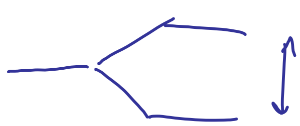
$$\vec{\mu}_{tot} = \vec{\mu}_{orb} + \vec{\mu}_{spin} = -\frac{e\hbar}{2m_e}(\vec{L} + 2\vec{S})$$

$\vec{\mu}_{orb}$: contributes to normal Zeeman effect

$\vec{\mu}_{spin}$: contributes to the anomalous Zeeman effect

example for $\ell = 0$, only $\vec{\mu}_{spin}$ contributes

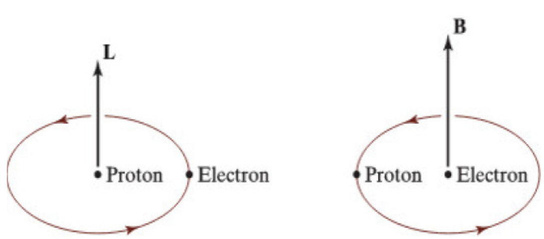
$$\Delta E = -\vec{\mu}_{tot}\cdot\vec{B} \underset{\vec{B} = B\hat{z}}{=} -\frac{e\hbar}{m_e}BS_z \Rightarrow S_z = \pm\tfrac{1}{2}\hbar \quad \Delta E = \pm\frac{e\hbar}{2m_e}B = \pm\mu_B B$$



$2\mu_B B$

$\Rightarrow$ even if no splitting from orbital normal Zeeman effect ($\ell = 0$), there is still a splitting in the energy (anomalous)

Fine structure



(a) (b)

e^- orbiting around the proton is proton's viewpoint. From e^-'s viewpoint proton is orbiting around it. Therefore

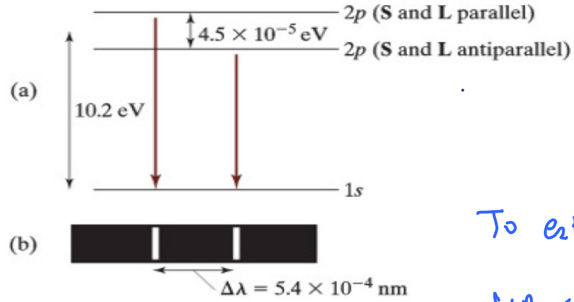
there is a B field produced by orbiting proton, which is proportional to electron's orbital angular momentum L

This gives energy contribution $\Delta E = -\vec{\mu}_{spin}\cdot\vec{B} = \frac{e}{m_e}\vec{S}\cdot\vec{B} \propto \vec{S}\cdot\vec{L}$ $\longrightarrow$ spin-orbit interaction

⇒ No contribution if $\ell = 0$

The splitting by spin-orbit interaction is small compared to $E_n = -\frac{E_R}{n^2}$

gives "fine structure" (between different n)



$\Rightarrow$ ΔE largest if $\vec{S} \parallel \vec{L}$

smallest " $\vec{S}$ anti-// $\vec{L}$

To estimate the separation from spin-orbit interaction we can just use $\Delta E = 2\mu_B B$, but need the value for B. From (problem 9.21) $B \approx 0.39\ T$, giving

$$\Delta E \approx 2 \times (5.8 \times 10^{-5}) \times 0.39 \approx 4.5 \times 10^{-5}\ eV$$

Using
$$\Delta\lambda = \frac{hc}{E_\gamma - \mu_B B} - \frac{hc}{E_\gamma + \mu_B B} \approx \frac{hc}{E_\gamma}\,\frac{2\mu_B B}{E_\gamma}$$

$$= \frac{1240\ nm \cdot eV}{10.2\ eV}\,\frac{4.5 \times 10^{-5}}{10.2} \simeq 5.4 \times 10^{-4}\ nm$$

9.21 •• The fine structure of an atomic spectrum results from the magnetic field "seen" by an orbiting electron. In this question you will make a semiclassical estimate of the B field seen by a $2p$ electron in hydrogen. The B field at the center of a circular current loop, i, of radius r is known to be $B = \mu_0 i/2r$. **(a)** Treating the electron and proton as classical particles in circular orbits (each as seen by the other), show that the B field seen by the electron is

$$B = \frac{\mu_0}{4\pi}\frac{eL}{m_e r^3} \qquad (9.37)$$

where L is the electron's orbital angular momentum ($L = m_e v r$ for a circular orbit). Remember that the current produced by the orbiting proton is $i = ev/2\pi r$, where v is the speed of the proton as seen by the electron (or vice versa). **(b)** For a rough estimate, you can give L and r their values for the $n = 2$ orbit of the Bohr model, $L = 2\hbar$ and $r = 4a_B$. Show that this gives $B \approx 0.39$ T and hence that the separation, $2\mu_B B$, of the two $2p$ levels is about 4.5×10^{-5} eV.

It should be clear that this semiclassical calculation is only a rough estimate. You have used the Bohr values for L and r. If, for example, you had used the quantum value $L = \sqrt{2}\hbar$, this would have changed your answer by a factor of $\sqrt{2}$. There is another very important reason that the argument used here is only roughly correct: The electron's rest frame is noninertial (since it is accelerated) and a careful analysis by the British physicist L. H. Thomas showed that the energy separation calculated here should include an additional factor of $\frac{1}{2}$. That our answer, 4.5×10^{-5} eV, is correct to two significant figures is just a lucky accident.

same magnitude as e^-, derived earlier

$$B = \mu_0 \frac{i}{2r} = \frac{\mu_0}{2r}\left(\frac{e/m_e\, L}{2A}\right)$$

$A = \pi r^2$

$$= \frac{\mu_0}{4\pi}\frac{eL}{m_e r^3}$$

$$\approx 10^{-7}\frac{N}{A^2}\cdot\frac{1.6\times10^{-19}\,\text{Coulb}\cdot 2\cdot 1.05\times10^{-34}\,\text{Js}}{9.11\times10^{-31}\,\text{kg}\,(4\cdot 0.53\times10^{-10}\,\text{m})^3}$$

$$= \frac{1.6\cdot 2\cdot 1.05}{9.11\cdot(0.53)^3}\cdot\frac{10}{4^3}\cdot\frac{N}{A^2}\,\frac{\text{Coulomb}}{\text{kg}}\,\frac{\text{J}\cdot\text{s}}{\text{m}^3}$$

$$\simeq 0.39 \text{ Tesla}$$

(We use SI units)