Relativistic mechanics

Galilean transform replaced by Lorentz transform.

Newtonian mechanics $\rightarrow$ relativistic mechanics (what is it?)

- time, position $\rightarrow$ velocity, acceleration $\rightarrow$ same def. (using appropriate ref. frame)

What about notation for mass? momentum? energy? how is force related to momentum and charge?

What do we expect of relativistic mechanics?

- valid in all inertial frames (invariant under Lorentz transform)
- reduces to non-relativistic one for $v \ll c$!
- agrees w. experiment

Mass: (b) $\Rightarrow$ same as classical $m = \frac{F}{a}$ valid for $v \ll c$! (cannot accelerate to larger $v$)

$\Rightarrow$ implies we should start from $v=0$ to apply $m = \frac{F}{a}$

i.e. bring object to rest before measuring mass (test mass, proper mass)

(this implies (a) will hold)

Momentum:

Newtonian $\vec{p} = m \vec{u}$ $\rightarrow$ it turns out with such a definition, conservation of momentum may not hold: $\sum \vec{p} = \text{const}$?

$\Rightarrow$ $\sum m \vec{u}_i$ conserved in frame S, but NOT in S'

Correct

relativistic $\vec{p} = m \frac{d\vec{r}}{dt}$ use proper time

$= m \frac{dt' \frac{d\vec{r}}{dt}}{dt} = m \gamma \vec{u}$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

before \((a,b)\) after \((-a,b)\)

\((-a,b)\)

\((a,b)\)
For an object with nonzero mass, its momentum will need to be very large when traveling close to speed of light!

For classical force \( F = ma = \frac{dp}{dt} \). It turns out that \( F = \frac{dp}{dt} \) is the correct definition in relativity!

If we use this definition of force and consider energy increase,

\[
dE = F \cdot dv = \frac{dp}{dt} \cdot dv = dp \cdot v = (m \gamma(w) u) \cdot v
\]

Let integrate along a straight line \( \vec{0} \rightarrow \vec{u} \)

\[
\int dE = \int (m \gamma(w) u) \cdot u = m \gamma(w) uu \bigg|_0^u - \int m \gamma(w) u du
\]

\[
E - E_0 = \frac{mu^2}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} \int_0^u \sqrt{1 - \frac{u^2}{c^2}} du = \frac{mu^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2
\]

\[
= m \frac{\sqrt{1 - \frac{u^2}{c^2}} + c^2 - u^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2 \Rightarrow E(u) - E(0) = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} - mc^2
\]

We identify \( E(u) \approx \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \approx \gamma mc^2 = \frac{dt}{dh} mc^2 \)

**Algorithm 2.10**

In Problem 2.14, it is proved that such energy expression gives conservation of energy in all inertial frames.

**When \( u=0 \), \( E = mc^2 \) (rest energy)**

Example 2.2: \( m = 1 \text{ kg}, \ E = mc^2 = 1 \text{ kg} \cdot (3 \times 10^8 \text{ m/s})^2 = 9 \times 10^{16} \text{ joules} \)

In practice we can only harness a small fraction of energy of power plant in one year!
Do we recover classical limit if \( \gamma \ll c \approx 1 \)?

\[
E (u) \approx mc^2 \left( 1 + \frac{\gamma^2 u^2}{c^2} \right) \approx mc^2 + \frac{1}{2} mu^2 \quad \text{yes!} \quad (mc^2 \text{ is just a constant})
\]

Kinetic energy \( K \equiv E - mc^2 = (\gamma - 1) mc^2 \).

**Example 2.3 Collision**

**Before** \( (m_1 \text{ and } m_2) \)

\[
\begin{array}{c}
\text{1} \\
\Theta \\
\text{u}_1 \\
\text{2} \\
\Theta \\
\text{u}_2 = 0
\end{array}
\]

**After**

\[
\begin{array}{c}
\text{1} \\
\Theta \\
\text{u}_3 \\
\text{2} \\
\Theta \\
\text{u}_4
\end{array}
\]

Assume conservation of momentum & energy (elastic collision)

Classical (Newtonian physics): \( u_3 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \), \( u_4 = \frac{2m_1}{m_1 + m_2} u_1 \)

Relativistic:

momentum: \( \gamma (u_1) m_1 u_1 = \gamma (u_3) m_1 u_3 + \gamma (u_4) m_2 u_4 \)

energy: \( \gamma (u_1) m_1 c^2 + m_2 c^2 = \gamma (u_3) m_1 c^2 + \gamma (u_4) m_2 c^2 \)

Algebra is quite messy and we obtain

\[
u_3 = \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2 + 2m_1 m_2 \sqrt{1 - u_1^2 / c^2}} u_1 \quad \text{→ reduces to classical one if } u \ll c \approx 1
\]

Two useful relations

\[
p = \frac{m u}{\sqrt{1 - u^2 / c^2}} \quad E = \frac{mc^2}{\sqrt{1 - u^2 / c^2}}
\]

\[
\frac{p}{E} = \frac{u}{c} = \frac{1}{\gamma c} \quad \text{for } \gamma = \frac{1}{\sqrt{1 - u^2 / c^2}}
\]

\( \gamma \to \infty \)

\[
E = \frac{mc^2}{1 - u^2 / c^2} = mc^2 \left( 1 + \frac{u^2 / c^2}{1 - u^2 / c^2} \right)
\]

\[
\Sigma E = (pc)^2 + (mc^2)^2
\]

\[
E = \frac{mc^2}{\sqrt{1 - u^2 / c^2}} \quad \text{(photon) \ (neutrino has mass!)}
\]

\[
\frac{p}{E} = \frac{u}{c} = \frac{1}{\gamma c} \quad \text{valid even for massless particle: } m = 0
\]

\[
E^2 = p^2 c^2
\]

\[
\text{i.e. } |u| = c \frac{p}{E} = \frac{c}{E} \quad \text{massless particles travel at speed of light!}
\]
For the collision problem:

\[ h_1 \sqrt{x_1^2 - 1} + 0 = m_1 \sqrt{x_2^2 - 1} + m_2 \sqrt{x_2^2 - 1} \]

\[
\begin{aligned}
&\quad (m_1 \sqrt{x_1^2 - 1} - m_1 \sqrt{x_2^2 - 1})^2 = m_2^2 (x_4^2 - 1) = m_2^2 \left( \frac{(m_1 y_1 + m_2 - m_1 y_3)^2}{m_2^2} - 1 \right) \\
&\quad = (m_1 y_1 + m_2 - m_1 y_3)^2 - m_2^2
\end{aligned}
\]

\[ m_1^2 (x_2^2 - 1) + m_1^2 (x_3^2 - 1) \]

\[ - 2 m_1 m_2 \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} = (m_1 y_1 + m_2 - m_1 y_3)^2 - m_2^2 \]

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \Rightarrow \quad \gamma^2 - 1 = \frac{\beta^2}{1 - \beta^2} = \beta^2 \]

\[ m_1^2 (x_2^2 - 1)^2 = m_1^2 (x_2^2 - 1) + \frac{1}{m_1} \left( \gamma_1 - \gamma_3 \right)^2 \]

\[ + 1 + \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} = \gamma_1 + m_2 \frac{\beta_1}{m_1} (\gamma_1 - \gamma_3) \]

\[ (\gamma_1^2 - 1)(\gamma_3^2 - 1) = \left( \frac{(-1 - \frac{m_2^2 y_1}{m_1}) (\gamma_1 - \frac{m_2}{m_1}) y_3}{m_2^2} \right)^2 \]

\[ = (\gamma_1 - \frac{m_2}{m_1})^2 y_3^2 - 2 (1 + \frac{m_2}{m_1}) (\gamma_1 - \frac{m_2}{m_1}) y_3 + \left( 1 + \frac{m_2^2}{m_1^2} \right) \gamma_3 \]

\[ \gamma_1 \gamma_3 + \beta^2 \]

Convenient to define:

\[ S = \frac{m_1}{m_2} \left( (y_1 + \gamma_1^2 - (y_1^2 - 1)) \gamma_3^2 - 2 (1 + \gamma_1) (y_1 + \gamma_1^2) y_3 + (1 + \gamma_1^2) - (y_1^2 - 1) \right) = 0 \]

\[ \Rightarrow \quad (2 y_1^2 + \gamma_1^2 + 1) y_3^2 - 2 (1 + \gamma_1) (y_1 + \gamma_1^2) y_3 + \gamma_3^2 + 2 \gamma_1 y_3 + \gamma_1^2 = 0 \]
Two solutions:
\[ \gamma_3 = \frac{\sqrt{1 + 2px + p^2}}{2x + p^2 + 1} (\gamma_1^2 + 2x\gamma_1 + \gamma_1) \]

\[ \gamma_3 = \frac{1 + 2px + p^2}{2x + p^2 + 1} (\gamma_1^2 + 2x\gamma_1 + \gamma_1) \]

So
\[ \gamma_3 = \frac{(1 + 2px + p^2)}{2x + p^2 + 1} (\gamma_1^2 + 2x\gamma_1 + \gamma_1) \]

\[ \gamma_3 = \gamma_1 \text{ or } \gamma_3 = \frac{\gamma_1^2 + 2x\gamma_1 + \gamma_1}{2x + p^2 + 1} \]

Given 
\[ \gamma_3 = \gamma_1 \]

\[ \beta_3 = 1 - \frac{(2px + p^2 + 1)}{(2x + p^2 + 1)^2} \]

\[ \beta_3 = \frac{\sqrt{\gamma_1^2 - 1}}{\gamma_1 (p^2 + 1) + 2x} \]

Hence
\[ U_3 = \frac{m_1^2 - m_2^2}{m_1^2 + m_2^2 + 2m_1m_2\sqrt{1 - \beta_3^2}} \]
Example 2.4 & 2.5 using eV & MeV as energy units

**Energy unit eV.** In addition to joule = N·m·m = kg·m²/s²

We can also use electron volt eV = (1.6×10⁻¹⁹ coulomb) x volt

= 1.6×10⁻¹⁹ Joule

MeV = 10⁶ eV = 1.6 × 10⁻¹³ J

Gev = 10⁹ eV = 1.6 × 10⁻¹⁰ J

Then mass can be measured in units of eV/c² = \( \frac{1.6 \times 10^{-19} J}{(3 \times 10^8 m/s)^2} \) = 1.738 \times 10^{-36} kg

**Example 2.4**

electron mass \( m_e = 9.109 \times 10^{-31} \) kg

\( = \frac{9.109 \times 10^{-31} \text{ eV}}{1.738 \times 10^{-36} \text{ c}^2} = 5.11 \times 10^5 \text{ eV/c}^2 \)

\( = 0.511 \text{ MeV/c}^2 \)

Physicists often just say electron’s mass is 0.511 MeV

(with \( \frac{1}{c^2} \) implicit)

In this unit, proton mass = 938.27 MeV/c²

neutron mass = 939.57 MeV/c²

**Example 2.5**

mass unit MeV/c² then momentum unit is MeV/c

If electron is moving with total energy \( E = 1.3 \text{ MeV} \)

momentum \( p = ? \) and speed \( u = ? \)

\( E = \sqrt{m_e^2 c^4 + p^2 c^2} \)

\( p = \sqrt{E^2 - (m_e c^2)^2} / c \) = \( \sqrt{(1.3)^2 - (0.5)^2} / c \) = 1.2 MeV/c

\( \frac{u}{c} = \beta = \frac{p c}{E} = \frac{1.2}{1.3} = 0.92 \)

\( \Rightarrow u = 0.92 c \)
Mass $\Rightarrow$ Energy

Process violates classical conservation of mass but logically necessary in relativity.

$m_1 \rightarrow \text{slowly from } \infty \rightarrow m_2 \Rightarrow \text{could have repulsive or attractive when close}

\[ m_1 \circ \text{ bring together } m_2 \rightarrow \text{need to put in energy (or work) to stabilize whole unit} \]

\[ \text{potential energy stored in the bound system as increase of rest mass: } M > m_1 + m_2 \]

\[ \text{total energy } = Mc^2 = k_1 + m_1 c^2 + k_2 + m_2 c^2 > (m_1 + m_2)c^2 \]

\[ k_1 + k_2 = c^2 \left[ M - (m_1 + m_2) \right] \text{ energy release} \]

\[ \Delta M = M - (m_1 + m_2) \]

\[ \text{i.e. mass } \Delta M \Rightarrow \text{converted to kinetic energy } k_1 + k_2 \]

\[ \text{(repulsive case) } \frac{\Delta Mc^2}{WR} \]

\[ \Delta M = m_1 + m_2 - M > 0 \]

\[ \Delta MA c^2 = \text{binding energy } B = \text{energy to pull bodies apart} \]

\[ = \text{energy released as bodies come to form bound state} \]

Example 2.6 - 2.8

2.11 w. massless particles
Example 2.6

The nuclei of certain atoms are naturally unstable, or radioactive, and spontaneously fly apart, tearing the whole atom into two pieces. For example, the atom called thorium 232 splits spontaneously into two "offspring" atoms, radium 228 and helium 4,

\[ {}^{232}\text{Th} \rightarrow {}^{228}\text{Ra} + ^{4}\text{He} \] (2.32)

The combined kinetic energy of the two offspring is 4 MeV. By how much should the rest mass of the "parent" \(^{232}\text{Th}\) differ from the combined rest mass of its offspring? Compare this with the difference in the measured masses listed in Appendix D.

<table>
<thead>
<tr>
<th>Atom</th>
<th>Mass (in u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{228}\text{Ra})</td>
<td>228.031</td>
</tr>
<tr>
<td>+ 4.003</td>
<td>232.034</td>
</tr>
</tbody>
</table>

\[ \Delta M = M_{\text{initial}} - (M_{\text{Ra}} + M_{\text{He}}) \]

1 u = \(1.66 \times 10^{-27}\) kg

(Carbon 12 atom mass \(= \frac{12.000000}{6.022 \times 10^{23}} = 1.99 \times 10^{-26}\) kg)

Example 2.7

It is known that two oxygen atoms attract one another and can unite to form an \(\text{O}_2\) molecule, with the release of energy \(E_{\text{out}} \approx 5 \text{ eV}\) (in the form of light if the reaction takes place in isolation). By how much is the \(\text{O}_2\) molecule lighter than two O atoms? If one formed 1 gram of \(\text{O}_2\) in this way, what would be the total loss of rest mass and what is the total energy released? (The \(\text{O}_2\) molecule has a mass of about \(5.3 \times 10^{-26}\) kg.)

\[ \beta = \frac{\Delta M c^2}{c^2} = \left(\frac{m_{\text{O}_2} - M_0}{c^2}\right) \approx 5 \text{ eV} \]

\[ M_{\text{O}_2} = 15.994 \text{ u} = 2.63 \times 10^{-26} \text{ kg} \]

\[ \Delta M = \frac{\beta c^2}{c^2} = 5 \text{ eV} = \frac{5 \times 1.778 \times 10^{-36} \text{ kg}}{c^2} \]

\[ M_{\text{O}_2} = 2 M_0 - \Delta M = 5.31 \times 10^{-26} \text{ kg} \]

\[ 1 \text{ g} \rightarrow \text{lose}\ 1.7 \times 10^{-10}\ \text{ g}\ \text{only} \]

\[ \Delta E = \frac{\Delta M c^2}{1.778 \times 10^{-36}} \text{ eV} \approx 10^{23} \text{ eV} \approx 1.6 \times 10^{4} \text{ J} \]

\(\text{cf. 2260 J to build 1 g of water}\)
Example 2.8

The Λ particle is a subatomic particle that (as mentioned in Example 1.2) can decay spontaneously into a proton and a negatively charged pion.

\[ \Lambda \rightarrow p + \pi^- \]

(This immediately tells us that the rest mass of the Λ is greater than the total rest mass of the proton and pion.) In a certain experiment the outgoing proton and pion were observed, both traveling in the same direction along the positive x axis with momenta

\[ p_p = 581 \text{ MeV}/c \quad \text{and} \quad p_\pi = 256 \text{ MeV}/c \]

Given that their rest masses are known to be

\[ m_p = 938 \text{ MeV}/c^2 \quad \text{and} \quad m_\pi = 140 \text{ MeV}/c^2 \]

find the rest mass \( m_\Lambda \) of the Λ.

\[ \Rightarrow \quad m_\Lambda \approx 1116 \text{ MeV}/c^2 \]

Example 2.11

As we will discuss in Chapters 17 and 18, there is a subatomic particle called the positron, or antielectron, with exactly the same mass as the electron (0.511 MeV/c^2) but the opposite charge. The most remarkable property of the positron is that when it collides with an electron, the two particles can annihilate one another, converting themselves into two or more photons. Consider the case that the electron and positron are both at rest and that just two photons are produced, with energies \( E_1 \) and \( E_2 \) and momenta \( p_1 \) and \( p_2 \) (Fig. 2.6). Use conservation of energy and momentum to find the energies \( E_1 \) and \( E_2 \) of the two photons.

Before: \( e^- (\bar{e}^+) \)

After: \( E_1, p_1 \quad E_2, p_2 \)

Momentum

\[ p_1 + p_2 = 0 \quad \Rightarrow \quad p_1 = -p_2 \]

Energy

\[ E_1 + E_2 = 2m_e c^2 \]

\[ 2p_2 c = 2m_e c^2 \quad \Rightarrow \quad |p_1| = |p_2| = m_e c = 0.511 \text{ MeV}/c \]

\[ E_1 = E_2 = 0.511 \text{ MeV} \]

Energy in the form of mass is converted to that in the form of electromagnetic waves!
Force again

\( F = \frac{dp}{dt} \) earlier we used this to derive energy \( E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \sqrt{p^2c^2 + m^2c^4} \)

Example 2.9 simply does the reverse: use energy expression \( E \) to show that \( dE = F \cdot dr \)

\[ \frac{dE}{dt} = \frac{1}{\beta} \sqrt{p^2c^2 + m^2c^4} = \frac{1}{2} \frac{d}{dt} \left( \frac{p^2c^2}{E} \right) \cdot c^2 = \frac{\beta^2 \cdot \frac{dp}{dt} \cdot c^2}{E} = \vec{u} \cdot \frac{dp}{dt} \]

\[ \downarrow \text{recall} \frac{\beta^2}{E} = \vec{u} \]

So \( \frac{dE}{dt} = \vec{u} \cdot \vec{F} = \frac{dp}{dt} \cdot \vec{F} \quad \Rightarrow \quad dE = \vec{F} \cdot d\vec{r} \quad \blacksquare \)

Lorentz force \( \vec{F} = q (\vec{E} + \vec{u} \times \vec{B}) \)

E.g. with only \( \vec{B} \), \( \vec{F} = q \vec{u} \times \vec{B} = \frac{dp}{dt} = m \frac{d}{dt} (\gamma u \vec{u}) \)

Since \( \vec{F} \) is perpendicular to \( \vec{u} \), \( u \)'s magnitude will not increase

One way to see it is to recall classical circular motion, force only changes magnitude (no work done) \( \Rightarrow \) speed remains the same

Similarly here \( \vec{F} \cdot \vec{u} = 0 \)

\[ \frac{dE}{dt} = 0 \quad E = \sqrt{p^2c^2 + m^2c^4} = \text{constant} \]

Thus, \( m \frac{d}{dt} (\gamma u \vec{u}) = m \gamma \frac{du}{dt} = q \vec{u} \times \vec{B} \)

Assume particle moves in a plane \( \perp \vec{B} \)

\( \Rightarrow \) \( m \gamma a = q \vec{u} \vec{B} \Rightarrow m \gamma \frac{u^2}{R} = q \vec{u} \vec{B} \Rightarrow \)

\[ R = \frac{mvu}{qB} = \frac{p}{qB} \]

or \( p = qB R \) momentum can be measured for charge particles!
Example 2.10

Proton momentum unknown \( p \)

\[
p = 8 \cdot B R = 1.6 \times 10^{-19} \times 1 \times 1.4 \text{ kg m/s} = 2.24 \times 10^{-19} \text{ kg m/s}
\]

\[
E = \sqrt{p^2 c^2 + m_p c^4} = \sqrt{420^2 + 938.27^2} \text{ MeV} = 1030 \text{ MeV}
\]

When is non-relativistic mechanics good enough?

It depends on the desired accuracy.

E.g., example. \( v = 300 \text{ m/s} \) for an hour, time dilation is several nanoseconds \( 1 \text{ ns} = 10^{-9} \text{ s} \)

But for application in GPS, nanosecond error affects accuracy of location e.g., meter

| TABLE 2.4 |

The relativistic and nonrelativistic kinetic energy of a mass \( m \) at various speeds \( u \), in units of \( mc^2 \).

<table>
<thead>
<tr>
<th>( u )</th>
<th>0.01c</th>
<th>0.1c</th>
<th>0.5c</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{rel} = (\gamma - 1)mc^2 ) [\text{rel}]</td>
<td>( 5.0004 \times 10^{-5} )</td>
<td>( 5.038 \times 10^{-3} )</td>
<td>0.155</td>
</tr>
<tr>
<td>( K_{NR} = \frac{1}{2}mu^2 ) [\text{NR}]</td>
<td>( 5.0000 \times 10^{-5} )</td>
<td>( 5.000 \times 10^{-3} )</td>
<td>0.125</td>
</tr>
<tr>
<td>% difference</td>
<td>0.01%</td>
<td>1%</td>
<td>20%</td>
</tr>
</tbody>
</table>

1. If \( v \ll 0.1c \) nonrelativistic mechanics ok.
2. If \( v \approx 0.1c \) should use relativity.
3. Massless particles are always relativistic.

\( \Leftrightarrow \) energy

0. \( K < 1\% \) of rest energy \( mc^2 \) \( \Rightarrow \) non-relativistic
1. \( K > 1\% \) \( mc^2 \) \( \Rightarrow \) relativistic
A few words on general relativity.

Presence of mass charge curvature of spacetime and mass moves in the geodesic of spacetime. Mathematics involves tensor analysis and is more complicated. We won't need general relativity in PH251. But it certainly is one of the most important discoveries in 20th Century.

Consequence of general relativity:

1. Bending of light by gravity

   ![Diagram of bending of light](image)

   (nowadays important tool in astrophysics such as gravitational lensing in intervening dark matter)

2. Gravitational Redshift

   ![Diagram of gravitational redshift](image)

3. Precession of Mercury's orbit

   ![Diagram of precession of Mercury's orbit](image)
4) Black Holes

5) Gravitational wave (already detected using Michelson interferometer in LIGO!)

GP-B was designed to measure two key predictions of Einstein's general theory of relativity by monitoring the orientations of ultra-sensitive gyroscopes relative to a distant guide star. Learn more about the mission.

https://einstein.stanford.edu/

GPS is an application of special & general relativity.

See Probs 2.47–2.49 for further analysis.