# Statistical Physics (Some selections)

Note Title 4/16/2017

In order to understand a system with a macroscopic number of particles, e.g. $10^{23}$, new theory is needed as it is impossible to keep track all particles. This is called statistical physics/mechanics, which can be applied to classical and quantum particles. This is a bridge from microscopic theory to thermodynamics.

## The Boltzmann factor (§15.3)

The most important quantity in statistical physics is the Boltzmann factor/weight:

For a system in equilibrium at a temperature $T$, the probability that it is in a (quantum) state $i$ with energy $E_i$ is proportional to $e^{-E_i/kT}$

$$P(\text{state } i) = C\, e^{-E_i/kT} \qquad (k \text{ is the Boltzmann const})$$

Using $\sum_j P(\text{state } j) = C \sum_j e^{-E_j/kT} = 1 \Rightarrow C = \dfrac{1}{\sum_j e^{-E_j/kT}}$

so
$$P(\text{state } i) = \frac{e^{-E_i/kT}}{\sum_j e^{-E_j/kT}}$$

$$Z \equiv \sum_j e^{-E_j/kT} \quad : \text{partition function}$$

$$P(E) = \sum_{\substack{\text{state } i \\ \text{with energy } E}} P(i) = g(E)\, \underset{\text{with } E}{\underset{\downarrow}{P(i)}} = \frac{g(E)\, e^{-E/kT}}{Z}$$

e.g. Use 52 cards as 52 quantum states and four different energies (♠ ♥ ♣ ♦)

$\Rightarrow$ each w. degeneracy $= 13$

$P(i) = \frac{1}{52}$. but $P(\heartsuit) = g(E)\, P(i) = 13 \cdot \frac{1}{52} = \frac{1}{4}$

## Example 15.3

Consider a gas of hydrogen in equilibrium at room temperature, $T \approx 293$ K. What is the ratio of the number of atoms in the first excited state to the number of atoms in the ground state?

$E_1 = -13.6\,\text{eV}$. $g(E_1) = 2$. 1s & ↑/↓ $\quad kT \approx 0.0252\,\text{eV}$

$E_2 = \frac{E_1}{4} = -3.4\,\text{eV} \quad g(E_2) = 2 \cdot 2^2 = 8$

$$\frac{P(E_2)}{P(E_1)} = \frac{g(E_2)\, e^{-E_2/kT}}{g(E_1)\, e^{-E_1/kT}} = \frac{g(E_2)}{g(E_1)}\, e^{-(E_2 - E_1)/kT} = 4 \cdot e^{-405} \approx 10^{-176}$$

⇒ hydrogen atom is mostly in lowest / ground state!

## Example 15.4

In hydrogen gas, at what temperature is the ratio of the number of atoms in the first excited state to the number of atoms in the ground state equal to 1/100; that is, how hot does hydrogen gas have to be for a significant fraction of the atoms to be in excited states?

$$\frac{P(E_2)}{P(E_1)} = 4 \cdot e^{\frac{10.2\,\text{eV}}{kT}} = \frac{1}{100}$$

$$\Rightarrow \frac{10.2\,\text{eV}}{kT} = \ln \frac{1}{400} \Rightarrow kT \approx 1.7\,\text{eV} \Rightarrow T \approx 20{,}000\,\text{K}!$$

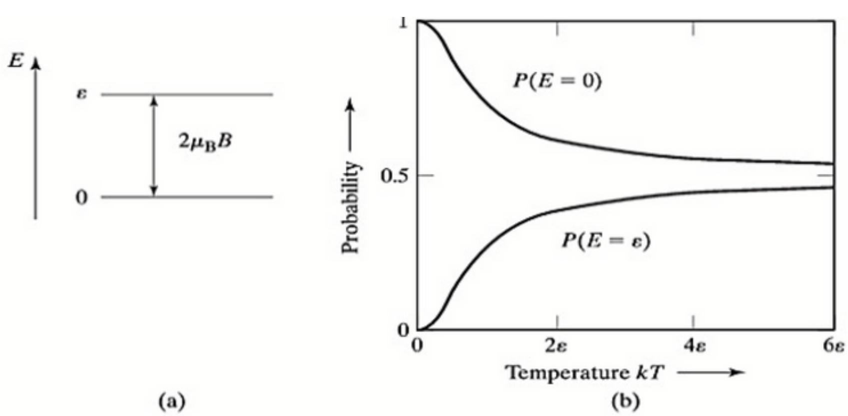



**FIGURE 15.3**
**(a)** The energy spectrum of an electron in a magnetic field, an example of a two-level system. **(b)** Probabilities that the electron is found in the ground state and in the excited state, as functions of $kT$.

## Example 15.5

Consider a particle with spin half and a magnetic moment $\boldsymbol{\mu}$ in an external magnetic field $\mathbf{B}$. Two orientations of the moment with respect to the field are allowed, corresponding to spin up and spin down. Find the probability that the moment is aligned with the field, in terms of the temperature $T$ and field $B$. What is this probability at room temperature (293 K) in a field of $B = 1.0$ T?

As described in Section 9.6, an electron in an external magnetic field has two energy states: a ground state with the moment aligned with the field and a first excited state with the moment anti-aligned with the field. The separation of the energy levels is $\varepsilon = 2\mu_B B$ where $\mu_B$ is the Bohr magneton. This is an example of a two-level system, a system with exactly two states, a ground state and an excited state. A two-level system is one of the few systems for which it is easy to write down the partition function (15.4) exactly. If we set the zero of energy at the ground-state energy, then the two levels have energies 0 and $\varepsilon$ [see Fig. 15.3(a)], and the partition function becomes

$$\sum_j e^{-E_j/kT} = 1 + e^{-\varepsilon/kT}$$

The probabilities of the two states are then, according to (15.3)

$$P(E = 0) = \frac{1}{1 + e^{-\varepsilon/kT}} \quad \text{and} \quad P(E = \varepsilon) = \frac{e^{-\varepsilon/kT}}{1 + e^{-\varepsilon/kT}} = \frac{1}{e^{+\varepsilon/kT} + 1}$$

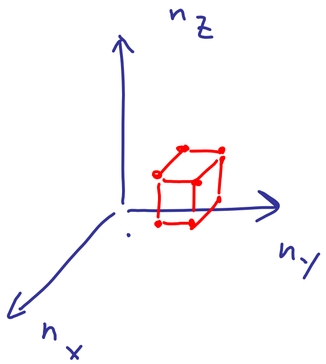
These two probabilities are plotted in Fig. 15.3(b). The upper curve is the probability $P(E = 0)$ that the system is in the ground state, with the magnetic moment aligned with the field; the lower curve is the probability that the system is in the excited state. In the high-temperature limit ($kT \gg \varepsilon$), both probabilities approach 0.5 and the system is equally likely to be in either state.

At $B = 1.0$ T, the energy-level separation is $2\mu B = (2)(5.79 \times 10^{-5}\ \text{eV/T})(1\text{T}) = 1.16 \times 10^{-4}$ eV, and the ground-state probability at $T = 293$ K ($kT = 0.0252$ eV) is

$$P(0) = \frac{1}{1 + e^{-\varepsilon/kT}} = \frac{1}{1 + e^{-2\mu B/kT}} = \frac{1}{1 + e^{-(1.16\times10^{-4})/(0.0252)}} = 0.5012$$

At room temperature, we are in the high-temperature regime ($kT \gg 2\mu B$) and the populations of the two levels are almost equal. There is a preference for the moments to be aligned with the B field, but the degree of alignment (or polarization) is only about 1 part in 500.

# Ideal gas (§15.7-8)




Free particles in a rigid box 3D:

$$E = \frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

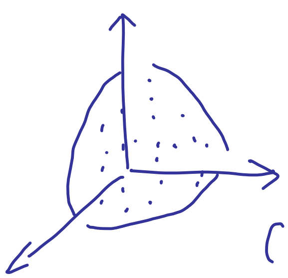
$$k_i = \frac{\pi}{a} n_i \qquad \Delta k_i = \frac{\pi}{a}\Delta n_i$$

$$d^3 n = \frac{V}{\pi^3}\, d^3 k$$

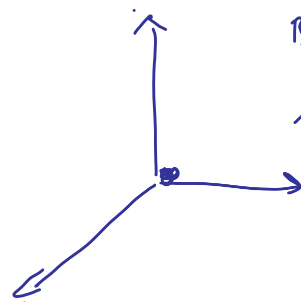
high temperature $\quad \frac{3}{2}kT = \frac{\hbar^2 k_{rms}^2}{2m}$ $\quad$ each particle has many unoccupied states to choose.

[ particle statistics not important ]

But at very low T



fermions fill up levels, similar to the periodic table (Pauli exclusion)



Bosons occupy lowest energy state

## energy & speed distribution in an ideal gas [w/o taking into particle statistics ]

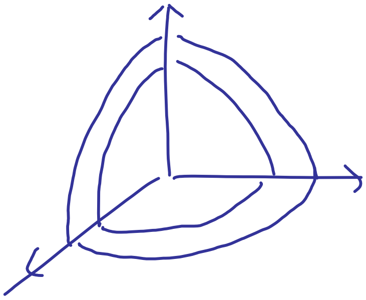
- treat points in $k$ space as a continuum

$\Rightarrow$ interested in

$$P(E)\,dE = \text{probability with energy in } (E, E+dE)$$

$$= \frac{D(E)\, e^{-E/kT}\, dE}{\int_0^\infty D(E)\, e^{-E/kT}\, dE}$$

$D(E)\,dE$ = # of states with energies in $(E, E+dE)$



$\#$ in $(k, k+dk)$ $\qquad d^3n = \dfrac{V}{\pi^3}\, d^3k \leftarrow 4\pi k^2 dk \times \dfrac{1}{8}$

$$= \frac{V}{2\pi^2} k^2 dk$$

consider <u>spinless</u> case & $E = \dfrac{\hbar^2 k^2}{2m}$ $\qquad k = \sqrt{\dfrac{2mE}{\hbar^2}}$

$$dE = \frac{\hbar^2 k}{m} dk$$

$$\Rightarrow D(E)\, dE = \frac{V}{2\pi^2} k^2 dk \Big| = \frac{V}{2\pi^2} \frac{2mE}{\hbar^2} \frac{m\, dE}{\hbar^2 \sqrt{\frac{2mE}{\hbar^2}}}$$

with $E = \dfrac{\hbar^2 k^2}{2m}$, $dE = \dfrac{\hbar^2 k}{m} dk$

$$D(E)\, dE = \frac{V}{2\pi^2} \cdot \sqrt{2}\, \frac{m^{3/2}}{\hbar^3} \sqrt{E}\, dE$$

$$\Rightarrow P(E)\, dE = \frac{e^{-E/kT} \sqrt{E}\, dE}{\int_0^\infty \sqrt{E}\, e^{-E/kT}\, dE}$$

$$\int_0^\infty \sqrt{x}\, e^{-x/b} dx = \frac{\sqrt{\pi b^3}}{2}$$

$$\boxed{P(E)\, dE = \frac{2}{\sqrt{\pi (kT)^3}} e^{-E/kT} \sqrt{E}\, dE}$$

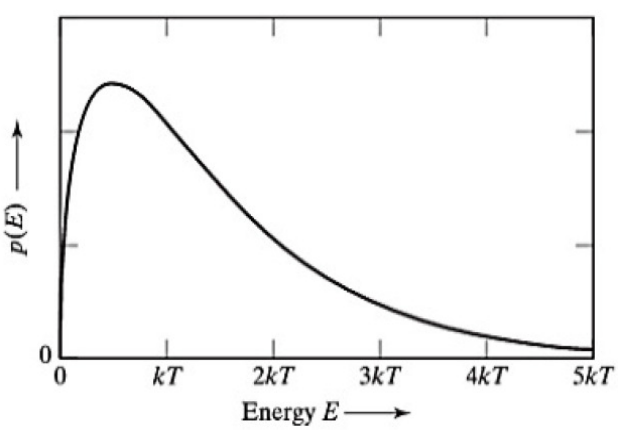


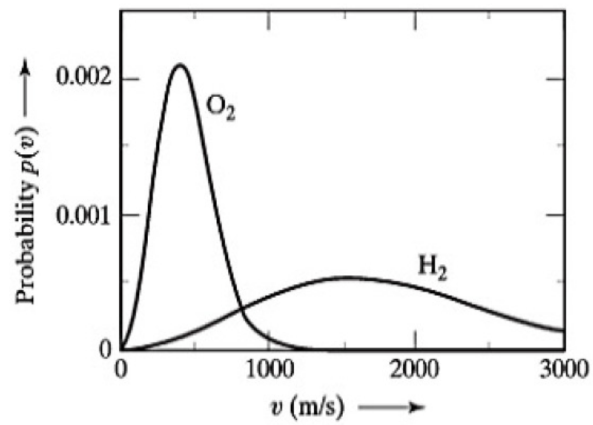

**FIGURE 15.14**
Distribution of single-particle energies in an ideal gas.

$$\boxed{P(E)\,dE = \frac{2}{\sqrt{\pi (kT)^3}}\, e^{-E/kT} \sqrt{E}\, dE}$$

in terms of $v$, $E = \frac{m}{2} v^2$

$$\frac{dE}{dv} = m v\, dv$$

$$p(v)\,dv = \frac{2}{\sqrt{\pi}\,(kT)^{3/2}}\, e^{-\frac{mv^2}{2kT}} \sqrt{\frac{m}{2}}\, m v^2\, dv$$




$$\boxed{p(v)\,dv = \sqrt{\frac{2}{\pi}} \left(\frac{m}{kT}\right)^{3/2} v^2\, e^{-\frac{mv^2}{2kT}}\, dv}$$

Maxwell speed distribution

With the two prob. distributions $\Rightarrow$ can calculate average values & peak location

$$\langle E \rangle = \int_0^\infty E\, p(E)\, dE = \frac{3}{2} kT$$

$$p'(E) = 0 \qquad \frac{1}{2}\frac{1}{\sqrt{E}} - \frac{1}{kT}\sqrt{E} = 0 \Rightarrow E_{peak} = \frac{kT}{2}$$

$$\langle v \rangle = \int_0^\infty v\, p(v)\, dv = \sqrt{\frac{8kT}{\pi m}}$$

$$p'(v) = 0 \qquad \left(2v - \frac{m}{kT} v^3\right) = 0 \Rightarrow v_{peak} = \sqrt{\frac{2kT}{m}}$$

probability distribution for a particle occupying a state with energy $E$

(av # per state)

① For classical particles

$$f(E) = e^{-(E-\mu)/kT}$$

$\mu$ : chemical potential

$\leftarrow$ from Boltzmann factor

② For bosons (Bose-Einstein distribution)

$$f_{BE}(E) = \frac{1}{e^{(E-\mu)/kT} - 1}$$

③ For fermions (Fermi-Dirac distribution)

$$f_{FD}(E) = \frac{1}{e^{(E-\mu)/kT} + 1}$$

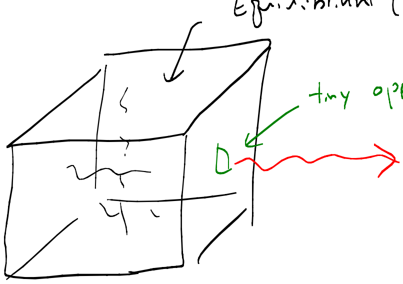
Black-body radiation (bosons : light/photons are players!)

Blackbody spectrum (* Example. cosmic microwave background radiation)

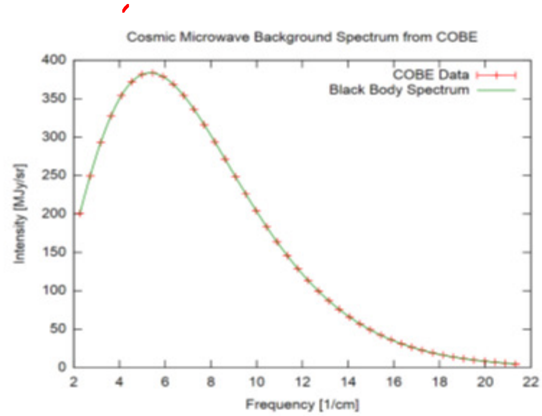
$T = 2.7260 \pm 0.0013$ K

Equilibrium (at T)

tiny opening



measure light (radiation)




① energy of a photon:

$$E = h\nu = \hbar\omega$$

② wave # $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

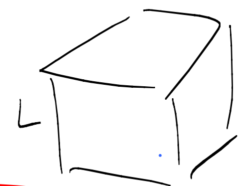
③ only two spin states ($m = +1$ or $-1$ but not 0) [2 polarizations]

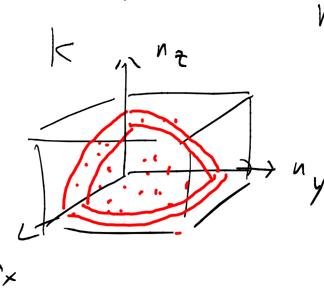
because of photons travel at speed of light $c$

⊕ # of photon is not conserved ⇒ $\mu = 0$

$$f(\hbar\omega) = \frac{1}{e^{\hbar\omega/k_BT} - 1}$$

$k_i = \frac{\pi}{L} n_i$ $\qquad n_i > 0$




For a box of volume $V = L^3$

$n_i = \frac{L}{\pi} k_i$

$d^3n = dn_x dn_y dn_z = d^3k \frac{V}{\pi^3}$

# of states

$$\boxed{d_k = 2\, d^3n = 2\left(\frac{L}{\pi}\right)^3 d^3k\, \frac{1}{8}}$$

$$= 2\frac{V}{\pi^3} 4\pi k^2\, dk \frac{1}{8} = \frac{V}{\pi^2} k^2\, dk = \frac{V}{\pi^2}\frac{\omega^2}{c^3}\, d\omega$$

⇒ energy density in $\omega$, $\omega + d\omega$

$$\frac{N_\omega \hbar\omega}{V} = \frac{\hbar}{\pi^2}\cdot\frac{\omega^3}{c^3}\frac{d\omega}{e^{\hbar\omega/k_BT} - 1} \equiv \rho(\omega)\, d\omega$$

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3\left(e^{\hbar\omega/k_BT} - 1\right)}$$

Planck's blackbody spectrum (energy per unit volume per unit freq)

$\omega t = 2\pi$

**4.1** • The **intensity distribution function** $I(\lambda, T)$ for a radiating body at absolute temperature $T$ is defined so that the intensity of radiation between wavelengths $\lambda$ and $\lambda + d\lambda$ is

$$(\text{intensity between } \lambda \text{ and } \lambda + d\lambda) = I(\lambda, T)\, d\lambda$$

This is the power radiated per unit area of the body with wavelengths between $\lambda$ and $\lambda + d\lambda$. The Planck distribution function for blackbody radiation is

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T} - 1} \qquad (4.28)$$

where $h$ is Planck's constant, $c$ is the speed of light, and $k_B$ is Boltzmann's constant. Sketch the behavior of this function for a fixed temperature for $0 < \lambda < \infty$. Explain clearly how you figured the trends of your graph. [*Hint:* You should probably think about the two factors separately.]

Power radiated per area

$c\,\rho(\omega)\, d\omega$

$= I(\lambda)\, d\lambda$

$\omega = \frac{2\pi c}{\lambda} \qquad d\omega = -\frac{2\pi c\, d\lambda}{\lambda^2}$

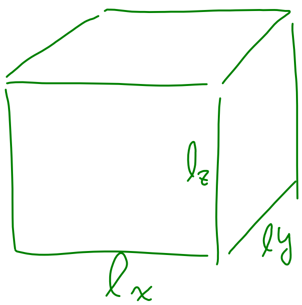
$$I(\lambda) = \frac{1}{4} c\, \rho(\omega) \left|\frac{d\omega}{d\lambda}\right|$$

$$= \frac{1}{4}\frac{h}{2\pi}\frac{c}{\pi^2 c^3}\frac{(2\pi c)^4}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

$$= \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$$

$$\frac{E_{Total}}{V} = \int_0^\infty \rho(\omega)\, d\omega \quad \text{(does not diverge!)} \quad = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4$$

Free electron gas. (Fermions: can apply to neutron star or white dwarf)




Assume in a infinite cubic box potential (0 inside, $\infty$ outside)

Energy level

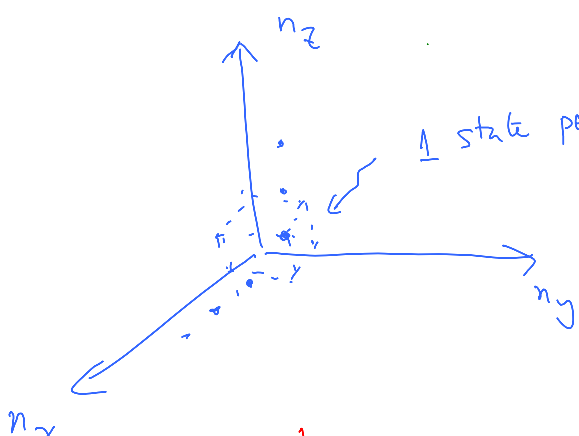
$$E_{n_x,n_y,n_z} = \frac{n_x^2\pi^2}{2ml_x^2} + \frac{n_y^2\pi^2}{2ml_y^2} + \frac{n_z^2\pi^2}{2ml_z^2} = \frac{\hbar^2\vec{k}^2}{2m}$$

w.f.

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{2}{l_x}}\sin\frac{n_x\pi x}{l_x}\sqrt{\frac{2}{l_y}}\sin\frac{n_y\pi y}{l_y}\sqrt{\frac{2}{l_z}}\sin\frac{n_z\pi z}{l_z}$$

$$n's = 1, 2, 3, \ldots$$

If there are $N$ electrons, they fill up the states (labeled by $(n_x, n_y, n_z)$)

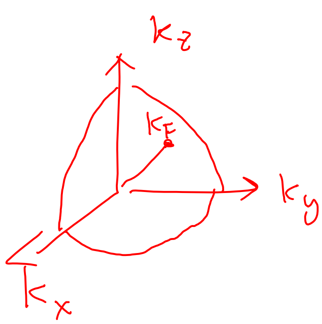



1 state per cube of volume 1 in $(n_x, n_y, n_z)$-space

or volume $\frac{\pi^3}{l_x l_y l_z} = \frac{\pi^3}{V}$ in $k$-space

Since energy is $\sim \vec{k}^2$

we occupy the volume up




to a radius $k_F$, which has

volume (in k) $\frac{1}{8} \times \frac{4\pi}{3}k_F^3$,

which contain # of states $= \frac{1}{8}\frac{4\pi}{3}k_F^3 \cdot \frac{V}{\pi^3}$

$= N$ (to accommodate all $e^-$) $/2$ ← $\frac{1}{2}$ comes from 2 spin states

$$\therefore\ k_F^3 = \frac{6\pi^2}{2}\frac{N}{V} = 3\pi^2\rho$$

$\rho$ → density

Define Fermi energy $E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3}$

Question: what is the total energy?

$$d_k = 2\times\frac{1}{8}\,4\pi k^2\, dk \cdot \frac{V}{\pi^3} = \frac{1}{\pi^2} k^2\, dk\, V$$

$$E_{total} = \int_{k>0}^{k_F} \underbrace{d^3k \times \left(\cdot \frac{V}{\pi^3}\right)}_{\text{how many orbitals}} \underbrace{\frac{\hbar^2 k^2}{2m}}_{\text{each contributes energy}} \times \underset{\substack{\uparrow \\ \text{two spin states}}}{2}.$$

[Note we are assuming zero temperature here.]

$$= \frac{1}{8} \cdot \int_0^{k_F} dk\, 4\pi k^2 \frac{V}{\pi^3} \frac{\hbar^2 k^2}{2m} \times 2$$

$$E_{total} = \frac{\hbar^2 V}{2\pi^2 m} \int_0^{k_F} dk\, k^4 = \frac{\hbar^2 V}{2\pi^2 m} \frac{k_F^5}{5} = \frac{\hbar^2 V}{10\pi^2 m} \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}\times 5}$$

$$= A\, V^{-2/3}$$ if you decrease volume, energy increases ⇒ pressure

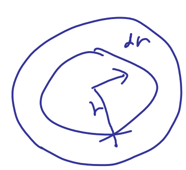
consider pressure $P = -\frac{\partial E_{total}}{\partial V} = \frac{2}{3} A V^{-\frac{5}{3}} = \frac{2 A V^{-\frac{2}{3}}}{3V} = \frac{2 E_{total}}{3V}$

⇒ degeneracy/exclusion pressure

prevents objects from collapsing

[Note we considered zero temperature, for finite temperature (like the black-body radiation in the photons) we need to include $n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1}$ ]

The above degeneracy pressure helps to stabilize neutron stars where gravitational pressure is large. To calculate that pressure, we start with gravitational energy:




$$E = -G \int_0^R \frac{dm\, M(r)}{r}$$

$$= -G \int_0^R \frac{(dr\, 4\pi r^2 \rho)\left(\frac{4}{3}\pi r^3 \rho\right)}{r} = -\frac{3}{5} G \frac{M^2}{R}$$

$M$ = total mass $= N m_N$

$$P_G = -\frac{3}{5} G M^2 \left(\frac{3V}{4\pi}\right)^{-1/3}$$ [ Need to write $\rho = \frac{N \cdot m_N}{V}$, fix N & $m_N$ ]

so $P_G = -\frac{dE}{dV} = \cancel{+} - \frac{1}{5} G M^2 \left(\frac{4\pi}{3}\right)^{1/3} V^{-\frac{4}{3}}$ $\quad M = N m_N$

Quoting the degeneracy pressure : $P_d = \frac{2}{3} \frac{\hbar^2}{10\pi^2 m_N} \left(3\pi^2 \frac{N}{V}\right)^{5/3}$

use neutron mass here

Balance the two pressures:

$$P_G + P_d = 0. \qquad \frac{1}{5} G N^2 m_N^2 \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-\frac{4}{3}} = \frac{2}{3} \frac{\hbar^2}{10\pi^2 m_N} (3\pi^2)^{5/3} N^{5/3} V^{-5/3}$$

$$\Rightarrow \quad V^{\frac{1}{3}} = \frac{2}{15} \frac{\hbar^2 \cdot 9\pi^2}{10 \cdot m_N^3} \left(\frac{4\pi}{3}\right)^{-1/3} N^{-\frac{1}{3}}$$

$$\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} R \quad \Rightarrow \quad \boxed{R = \frac{9 \hbar^2 \pi^2}{75 m_N^3} \left(\frac{4\pi}{3}\right)^{-2/3} N^{-\frac{1}{3}}}$$ radius of neutron star