Special Holonomy Metrics and Hitchin’s Functionals

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Stable Forms

Stable forms are defined as follows [1]. Let X be a manifold of real dimension \(n\), and \(V = TX\). Then, the form \(\rho \in \Lambda^p V^*\) is stable if it lies in an open orbit of the (natural) \(GL(V)\) action on \(\Lambda^p V^*\). In other words, this means that all forms in the neighborhood of \(\rho\) are \(GL(V)\)-equivalent to \(\rho\). This definition is useful because it allows one to define a volume. For example, a symplectic form \(\omega\) is stable if and only if \(\omega^{n/2} \neq 0\).

Here we give another example, for \(n = 7\) and \(p = 3\):

\[
\begin{align*}
\dim GL(V) &= n^2 = 49 \\
\dim \Lambda^p V^* &= \frac{n!}{p!(n-p)!} = 35 \\
14 &= \dim(G_2) \quad (1)
\end{align*}
\]

Volume Functionals

**Basic Idea:** For a stable \(p\)-form \(\rho\) on a compact manifold \(X\), construct a volume functional,

\[
V(\rho) = \int_X \phi(\rho)
\]

such that the critical points of \(V(\rho)\) define metrics of reduced holonomy, see Figure 1.

\[
\begin{align*}
V(\sigma) &= \int_X |\sigma|^\frac{4}{n} \\
n &= 6 \text{ and } p = 4:\ \\
V(\rho) &= \int_X \sqrt[n]{-\frac{1}{6} K_a^b K_b^a}, \quad (3)
\end{align*}
\]

where

\[
K_a^b := \frac{1}{12} \rho_{a_1 a_2 a_3} \rho_{a_4 a_5 a_6} \epsilon^{a_1 a_2 a_3 a_4 a_5 a_6}.
\]

Hamiltonian Flow

**Basic Idea:** For a (non-compact) manifold \(X\) foliated by a manifold \(Y\), construct a Hamiltonian system, with Hamiltonian \(H(x_i, p_i)\), such that the Hamiltonian flow equations

\[
\begin{align*}
\frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i} \\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i}
\end{align*}
\]

define a reduced holonomy metric on \((t_1, t_2) \times Y\).

Consider, for example, a homogeneous quotient space

\[
Y = G/K,
\]

where \(G\) is some group and \(K \subset G\) is a subgroup. Therefore, we can think of \(X\) as being foliated by principal orbits \(G/K\) over a positive real line, \(\mathbb{R}_+\), as shown on Figure 2. \(G/K\) may collapse into a degenerate orbit:

\[
B = G/H
\]

where symmetry requires

\[
G \supset H \supset K \quad (7)
\]

Moreover,

\[
H/K = S^k \Rightarrow X \text{ smooth}
\]

also implies

\[
X \cong (G/H) \times \mathbb{R}^{k+1} \quad (8)
\]

There exists a symplectic structure on the space, \(\mathcal{P}\), of \(G\)-invariant forms on \(Y = G/K [2]\):

\[
\mathcal{P} = \text{Phase Space} \quad \omega = \sum dx_i \wedge dp_i \quad (9)
\]

Moreover, there is a canonical construction of a Hamiltonian \(H(x_i, p_i)\) for our dynamical system, such that the Hamiltonian flow equations (4) are equivalent to the special holonomy condition [2].

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An Example

Let us take

\[
G = SU(2)^3 \quad H = SU(2)^2 \quad K = SU(2)
\]

From (5) it follows

\[
Y = SU(2) \times SU(2) \cong S^3 \times S^3
\]

Furthermore, \(G/H \cong H/K \cong S^3\) implies that \(X\) is a smooth manifold with topology, cf. (8),

\[
X \cong S^3 \times \mathbb{R}^4
\]

In order to find a \(G_2\) metric on this manifold, we need to construct the “phase space”, \(\mathcal{P}\), that is the space of \(SU(2)^3\)-invariant 3-forms and 4-forms on \(Y = G/K\):

\[
\mathcal{P} = \Omega^3_G(G/K) \times \Omega^4_G(G/K)
\]

It turns out that each of the factors is one-dimensional, generated by a 3-form \(\rho\) and by a 4-form \(\sigma\), respectively,

\[
\begin{align*}
\rho &= \sigma_1 \sigma_2 \sigma_3 - \Sigma_1 \Sigma_2 \Sigma_3 + x \left( d(\sigma_1 \Sigma_1) + d(\sigma_2 \Sigma_2) + d(\sigma_3 \Sigma_3) \right), \\
\sigma &= p^{3/2} \left( \sigma_2 \Sigma_2 \sigma_3 \Sigma_3 + \sigma_3 \Sigma_3 \sigma_1 \Sigma_1 + \sigma_1 \Sigma_1 \sigma_2 \Sigma_2 \right).
\end{align*}
\]

where \(\sigma_a\) and \(\Sigma_a\) are left invariant 1-forms:

\[
d\sigma_a = -\frac{1}{2} \epsilon_{abc} \sigma_b \wedge \sigma_c,
\]

\[
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\]

Therefore, we have only one “coordinate” \(x\) and its conjugate “momentum” \(p\), parametrizing the “phase space” \(\mathcal{P} = \Omega^3_{\text{exact}}(Y) \times \Omega^4_{\text{exact}}(Y)\) of our model. The non-degenerate symplectic structure looks like

\[
\omega((\rho_1, \sigma_1), (\rho_2, \sigma_2)) = \langle \rho_1, \sigma_2 \rangle - \langle \rho_2, \sigma_1 \rangle,
\]

where, in general, for \(p = d\beta \in \Omega^k_{\text{exact}}(Y)\) and \(\sigma = d\gamma \in \Omega^{n-k}_{\text{exact}}(Y)\) one has a non-degenerate pairing

\[
\langle \rho, \sigma \rangle = \int_Y \beta \wedge \gamma = (-1)^k \int_Y \beta \wedge d\gamma.
\]

Once we have the phase space \(\mathcal{P}\), it remains to define the Hamiltonian [2]:

\[
H = 2V(\sigma) - V(\rho),
\]

where \(V(\rho)\) and \(V(\sigma)\) are the volume functionals (3) and (2), respectively. Evaluating (16) for the \(G\)-invariant forms (12) and (13) we obtain the Hamiltonian flow equations:

\[
\begin{align*}
\dot{p} &= x(x - 1)^2 \\
\dot{x} &= p^2
\end{align*}
\]

The solution for \(x(t)\) and \(p(t)\) determines the evolution of the forms \(\rho\) and \(\sigma\) which, in turn, define the associative three-form on the 7-manifold \(Y \times (t_1, t_2)\),

\[
\Phi = dt \wedge \omega + \rho,
\]

where \(\omega\) is a 2-form on \(Y\), such that \(\sigma = \omega^2/2\). The associative form \(\Phi\) is automatically closed and co-closed,

\[
d\Phi = 0 \\
d \ast \Phi = 0
\]

Therefore, it defines a \(G_2\) holonomy metric on \(X\), viz. the \(G_2\) metric on the spin bundle over \(S^3\), originally found in [3, 4]. More examples can be found in [5–7].

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