Formalism

We want to use the divergence theorem to arrive at a formula for the divergence, i.e., \( \nabla \cdot \mathbf{A} \).

\[ \int_V \nabla \cdot \mathbf{F} = \int_{\text{surf}} \mathbf{F} \cdot d\mathbf{S} \quad \text{(1)} \]

It is important to discuss the direction of vectors taken in this convention particularly on the right hand side of this equation.

Consider:

\[ d\mathbf{S} \]

Note: These are 1 pair of opposite walls of a parallelepiped. There are 2 other pairs.

\( d\mathbf{S} \) is taken to be in the direction of the outward normal vector.

In this case, the outward normal vectors from both sides point away from each other.

Here, the difference between the two is the difference in the \( x \) direction.
The vector $\mathbf{F}$ is always taken to be in the direction of the outward normal as well. That is, it is the component of $\mathbf{F}$ in the outward direction.

Now we can take a volume. Let's take an infinitesimal volume of rectangular shape, or a parallelepiped.

We will attempt to find an expression for $\nabla \cdot \mathbf{F}$.

Starting from Gauss's theorem: It states that if you have a volume, $V$, that is surrounded by a surface (must be closed!), $S$, then if you have functions $G$, $H$, and $F$, such that

$$G(x, y, z), \quad H(x, y, z), \quad F(x, y, z)$$

and the derivatives of these functions

$$\frac{\partial G}{\partial x}, \quad \frac{\partial H}{\partial y}, \quad \frac{\partial F}{\partial z}$$

are continuous on the volume, on the surface, or on both $V$ and $S$, then

$$\int_V \left( \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial F}{\partial z} \right) dV = \int_S [G \cos(\hat{n}, x) + H \cos(\hat{n}, y) + F \cos(\hat{n}, z)] dS \tag{2}$$

Lec 4 (2)
(for proof of (2), see Vector and Tensor Analysis With Applications, by A.I. Borisenko and I.E. Tarapov pg 138-139)

Now let's suppose that we have a point, \( P_0 \), in our rectangular parallelepiped of volume \( V \),

\[ \boxed{P_0} \]

A way of measuring the strength of sinks or sources within our volume would be to use the right side of the divergence theorem, equation (1)

\[ \int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = \int_{V} \nabla \cdot \mathbf{F} \, dV \quad (3) \]

If we can make the volume of our parallelepiped go to zero, while keeping \( P_0 \) contained by the volume, we can measure the source strength of the point \( P_0 \). To accomplish this, divide the right side of (2) by the volume, and take the limit as \( V \to 0 \)

Lec 4 ③
\[ \lim_{V \to 0} \frac{1}{V} \oint_{S} \vec{f} \cdot d\vec{s} \quad \text{(4)} \]

\( \text{(4)} \) is the source strength of \( P_{0} \) if the limit exists. This is the divergence of \( \vec{f} \),

\[ \nabla \cdot \vec{f} = \lim_{V \to 0} \frac{1}{V} \oint_{S} \vec{f} \cdot d\vec{s} \quad \text{(5)} \]

This definition works for any coordinate system. The divergence only exists for a vector field at points where the components of the vector field and their derivatives are continuous.

We can now use Gauss's theorem, \( \text{(2)} \) to get the following result,

\[ \frac{1}{V} \oint_{S} \vec{f} \cdot d\vec{s} = \frac{1}{V} \int_{V} \left( \frac{\partial f_{1}}{\partial x} + \frac{\partial f_{2}}{\partial y} + \frac{\partial f_{3}}{\partial z} \right) dV \quad \text{(6)} \]

Now, we take the limit on both sides,

\[ \lim_{V \to 0} \frac{1}{V} \oint_{S} \vec{f} \cdot d\vec{s} = \lim_{V \to 0} \frac{1}{V} \int_{V} \left( \frac{\partial f_{1}}{\partial x} + \frac{\partial f_{2}}{\partial y} + \frac{\partial f_{3}}{\partial z} \right) dV \quad \text{(7)} \]
The right side of (7) becomes
\[ \frac{df_1}{dx} + \frac{df_2}{dy} + \frac{df_3}{dz} \]

Using (6), then, we get the formula for the divergence of \( f \) in rectangular coordinates,
\[ \nabla \cdot f = \frac{df_1}{dx} + \frac{df_2}{dy} + \frac{df_3}{dz} \quad (8) \]

(for more, see Vector Analysis, Homer E. Newell Jr.)

A different approach to this problem can be taken, still utilizing an infinitesimal rectangular volume \( V \). We have
\[ dx \wedge dy \wedge dz \quad \nabla \cdot f = dy \wedge dz \wedge dx \frac{df_x}{dx} \]
\[ + dz \wedge dx \wedge dy \frac{df_y}{dy} \]
\[ + dx \wedge dy \wedge dz \frac{df_z}{dz} \quad (9) \]
The symbol \( \Lambda \) is called a "hook," "hat," "wedge," or "exterior product sign." It is used to demonstrate an antisymmetrized tensor product (see *Gravitation*, Misner, Wheeler, Thorne, pg 83). This \( \Lambda \) is used to show that volume and surface elements are antisymmetric in \( dx, dy, dz \). If any two of these differentials were in the same direction, then there would be an inherent degeneracy, and \( \Theta \) would be \( 0 = 0 \).

The order in which these products are taken is important. The right side of \( \Theta \) obeys a cyclic order.

From \( \Theta \),

\[
\nabla \cdot \vec{E} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}
\]

**Magnetostatics continued:** Consider the following arrangement,

![Diagram](image)

\( I_1 \cdot B_1 \cdot \)
Both currents are running parallel to each other, with accompanying $\overrightarrow{B}$ fields acting on each other. We want to examine the force that acts between the wires.

The force on 2 by 1

$$\overrightarrow{F}_{\text{on}2} = \overrightarrow{J}_2 \times \overrightarrow{B}_1 (r)$$

Then the force direction is toward $I_1$. The magnitude of the force is given by

$$\overrightarrow{F}_{\text{on}2} = I_2 \hat{z} \times \frac{\mu_0 I_1 \phi}{2\pi r}$$

$$\Rightarrow \quad \frac{\overrightarrow{F}}{l} = -\frac{I_2 I_1 \mu_0}{2\pi r} \hat{r} \quad \text{Force per unit length}$$

$\overrightarrow{J} \times \overrightarrow{B}$ is the magnetostatic contribution to the force density equation,

$$\overrightarrow{F} = \rho \overrightarrow{E} + \overrightarrow{J} \times \overrightarrow{B}$$
Week 2  Lecture 5  (Sept. 7, 2001)

**Review of statics**

By looking at the div $\mathbf{E}$,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1)$$

we see that electric fields emanate from charges, and conversely,

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

that the magnetic field cannot come from a charge, i.e. no magnetic monopoles.

An examination of the curl equations of the two fields reveals that our outlook on the situation may be slightly askew,

$$\nabla \times \mathbf{E} = 0 \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (4)$$

Something needs to be added, at least to (4). We need to look at the force laws,

$$\mathbf{F} = q (\mathbf{E} + \nabla \times \mathbf{B})$$

$$\mathbf{F} = m \mathbf{a}$$

Lec5  ①
When we look at the expression for the \( \mathbf{E} \) field,

\[
\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{a}}{|\mathbf{r} - \mathbf{a}|^3}
\]

we see that there is no explicit time dependence. But, if a charge moves, we know from experiment that the \( \mathbf{E} \) field acquires a time dependence. We look at Newton's action at a distance law

\[
\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{a}(t)}{|\mathbf{r} - \mathbf{a}(t)|^3}
\]

Now the time dependence is explicit. Using equations 1-4, one can predict the motion of charged particles at low velocities. This was the state of electromagnetism from 1820-1832.

Then along came Faraday. Through exhaustive experimentation, he was able to show

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

i.e., a \( \mathbf{B} \) field that changes in time will induce an electric field.

LeC5 2
Faraday was able to show this by producing an emf in a circuit by allowing a time changing magnetic field to go through the area of the circuit

$$\mathcal{E}_{\text{circuit}} = -\frac{d\Phi}{dt}$$

To check the validity of the curl equation, we have to make sure that the $(\text{div})(\text{curl})\vec{E}$ is 0. This is a vector field rule,

$$\Rightarrow \nabla \cdot \nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \cdot \vec{B} = 0$$

To look at this situation, we examine a surface

Now, we look at Stokes' theorem with the $\vec{E}$ field,

$$\int_{\Sigma} \nabla \times \vec{E} \cdot d\Sigma = \oint_{\partial \Sigma} \vec{E} \cdot d\vec{l}$$
If we take the contribution of the flux of lines coming through surfaces I and II, and apply it to equation 6,

\[
\int_{\Sigma I} - \int_{\Sigma II} = \int_{\Sigma} \mathbf{dS} \cdot \nabla \times \mathbf{E} = 0
\]

⇒ net flux out of any volume is zero.

Faraday's law obeys this because

\[ \nabla \cdot \mathbf{B} = 0 \]

Let's look at equation 4

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]

Now, we know that the following has to be true

\[ \nabla \cdot \nabla \times \mathbf{B} = 0 \]

Let's see,

\[ \nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{J} \]

Lec 5  ④
Now let's look at the equation of continuity,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]

This gives

\[ \nabla \cdot \nabla \times \mathbf{B} = -\mu_0 \frac{\partial \rho}{\partial t} \]

There is a discrepancy between the left and right parts of this equation. Even if there is a current density, it is implied that the charge density cannot change in time \( \rightarrow \) this cannot be true!

Maxwell looked to circumvent this problem. Start by looking at the integral form for the case of a moving charged particle

\[ \int \mathbf{B} \cdot d\mathbf{A} \]

Now examine the following figure

Lec 5  5
If the current is independent of the surface, then there is a contradiction if \( 0 = \mu_0 \frac{dp}{dt} \) is used. This occurs because the charge density will change only as the charge penetrates the surface. Outside the surface, the moving charge has no effect on the B field. This does not seem right.

Let's see if we can fix this. We start by rewriting equation 0:

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \\
\Rightarrow \rho = \varepsilon_0 \nabla \cdot \vec{E} \\
\Rightarrow \frac{d\rho}{dt} = \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t}
\]

Now we look at \( \nabla \times \vec{B} \). Faraday obtained non-zero \( \nabla \times \vec{E} \), showing that it came from a time dependent \( \vec{B} \) field. Let's see if a time dependent \( \vec{E} \) field has an effect on \( \nabla \times \vec{B} \):

\[
\nabla \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})
\]

Lec 5 (c)
The term $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ is called the displacement current, and was added by Maxwell. To see how this helped, we take the $\nabla \cdot \nabla \times \vec{B}$

$$
\nabla \cdot \nabla \times \vec{B} = \mu_0 \left( \nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} \right)
$$

$$
= \mu_0 \left( -\frac{\partial \Phi}{\partial t} + \varepsilon_0 \frac{\partial}{\partial t} \frac{\vec{P}}{\varepsilon_0} \right)
$$

$$
= 0
$$

So, the displacement current solves our problem. The $\nabla \cdot \nabla \times \vec{B} = 0$, as it should be.

Now we have Maxwell's equations

$$
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad (5)
$$

$$
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (6)
$$

$$
\nabla \cdot \vec{B} = 0 \quad (7)
$$

$$
\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \quad (8)
$$

We consider now the situation that occurs when there are no charges or currents in space. Then (6) and (8) become

$$
\nabla \cdot \vec{E} = 0 \quad (9)
$$

$$
\nabla \cdot \vec{B} = 0 \quad (10)
$$

Lec5  (7)
and (1) and (8) become
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \text{unchanged} \quad (1) \]
\[ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (12) \]

Our goal is to determine the speed of light. First, take the curl of (11)
\[ \nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \times \vec{B} \quad (13) \]

Use the vector operator relation
\[ \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \]

With this, and (12) for \( \nabla \times \vec{B} \)
\[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \frac{\partial}{\partial t} \left( \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0 \]

but, \( \nabla \cdot \vec{E} = 0 \) from our assumptions
\[ \Rightarrow \nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (14) \]

This is a vector wave equation. We made the assumption in class that we were dealing with plane waves.
For a wave equation
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{V^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \]

Lec 5 (8)
$V$ is the velocity. In (14)

$$V^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$\Rightarrow V = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad \Rightarrow \text{should be } c$$

Check this value: $\varepsilon_0 = \frac{1}{4\pi (9 \times 10^9)}$

$\mu_0 = 4\pi \times 10^{-7}$

$$\Rightarrow V = \sqrt{\frac{1}{4\pi (9 \times 10^9)}} \cdot 4\pi \times 10^{-7}$$

$$V = \sqrt{\frac{9 \times 10^9}{10^{-7}}} = \sqrt{9 \times 10^{-16}}$$

$$\Rightarrow V = 3 \times 10^8 \frac{m}{s} = c$$

$\rightarrow \text{speed of light}$