Friday's high point was repeating Maxwell's calculation:

\[ (k^2 - \omega^2 \varepsilon_0 \mu_0) = 0 \]

\[ \Rightarrow \frac{\omega^2}{k^2} = \frac{1}{\varepsilon_0 \mu_0} \quad \varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} \]

So, \[ \varepsilon_0 \mu_0 = \frac{1}{V^2} = \frac{4\pi \times 10^{-7}}{(4\pi \times 9 \times 10^9)} = 9 \times 10^{-16} \]

\[ \Rightarrow V = 3 \times 10^8 \text{ m/s} ! ! \]

So, maybe light is an E.M. wave. However, it was decades until Hertz demonstrated this.

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Our next topic will begin the high point of this semester: Einstein's Special Theory of Relativity.

2 keys to Relativity:

1. 1905 Einstein's C.V.
   1) Brownian Motion
   2) P.E. effect
   3) S.T.R

2. Principle of Relativity
   The laws of physics take the same form in any 2 frames of reference moving w.r.t. each other at constant velocity.

III) \( c \) is a universal constant.

By 1904, Lorentz, Fitzgerald, Poincaré were all at the threshold but they could not relinquish their preferred reference frame.
What kind of relationship can we find between $e^{i\theta}$ consistent with Galilean relativity?

$x' = x + vt$
$t' = t$

We will borrow some quantum mechanical techniques.

General Sh. Wave Eq. $H\psi = i\hbar \frac{\partial \psi}{\partial t}$

$t = 1 c = 1 \ [i\hbar \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t}]$

$[\psi, x] = -i$

Conceptual framework to understand what we will do.

Think of $|\psi(x, t)|^2$ in general $|\psi(x, t)|$ could be anything for one frame moving w.r.t. other.

we know $|\psi(x', t')|^2$ must $|\psi(x, t)|^2$ for one frame.

Think of a wave packet in the rest frame.

at some later time it is still in the same place but its shape changes in the moving frame it looks like.
How will $H$ and $p$ change?

$\Psi(x', t') = e^{i(H, x', t') \Psi(x - dV/t, t)}$

$\Psi(x', t')$ is the phase factor.

$H' = H + dH$

$dH$ (in general) = $p dV + \text{const} dV + x dV$

$\checkmark$

1. Vector $\cdot$ Vector $\rightarrow$ Scalar that's good
   also $[H, p] = 0$

2. Const $\cdot$ Vector $\rightarrow$ Vector, $dH$ is a scalar + we must maintain rotational invariance

3. Vector $\cdot$ Vector $\rightarrow$ Scalar that's good
   BUT $[H', p]$ would no longer equal zero

   How about $\tilde{p}$?

   $\tilde{p}' = \tilde{p} + d\tilde{p}$
   $d\tilde{p} = H dV + \tilde{p} dV + \text{const} x + \text{const}$

4. Scalar $\cdot$ Vector $\rightarrow$ Vector, good
   BUT it would require $\Psi(x', t') = \Psi(x - dV/t, t - AdV x)$ this violates Galilean rel $\rightarrow t = t'$

$\checkmark$

5. Vector $\cdot$ Vector $\rightarrow$ Scalar

6. Scalar $\cdot$ Vector $\rightarrow$ Vector but $[H, p'] \neq 0$
\[ \Delta H, \psi \Rightarrow 0 \quad \text{good} \]

\[ \text{Choose } |\psi_0\rangle \text{ to be a basis for primed and unprimed systems} \]

\[ \rho |\psi_0\rangle = 0 |\psi_0\rangle \quad \text{particle at rest} \]

\[ \rho' |\psi_0\rangle = |\psi_0\rangle + MV |\psi_0\rangle \]

\[ \rho |\psi_0\rangle = E_0 |\psi_0\rangle \]

\[ H |\psi_0\rangle = E_0 |\psi_0\rangle + \int V' dV' \rho(V') \]

\[ \int V' dV' MV = \frac{1}{2} MV^2 \]

\[ H' |\psi_0\rangle = E_0 |\psi_0\rangle + \frac{1}{2} MV^2 \]

\[ H = \frac{p^2}{2m} \quad \text{for Galilean relativity} \]

\[ \text{E}_{\text{e}} \text{can be set equal to zero in the real world it may exist but we can't always measure it.} \]
How do we reconcile principle II with principle I?

First try Galilean:

\[ v' = v + v' \]

but for light: \( c' = c + v' \) which violates principle II.

Before Einstein's bold move, there was postulated some special frame that can't be detected.

How can we repair this situation?

Imagine \( x' = f(x, t) \) is too general. \( t' = g(x, t) \) Einstein figured out how to restrict \( f + g \).

Einstein tried to imagine riding a light wave crest. Can't be done because light can't be at rest.

Translation invariance has to be respected:

\[ \begin{align*}
    dx & \rightarrow dx', dt' \quad \text{independent} \\
    dt & \rightarrow dx', dt' \quad \text{of } x+t \text{ or origin}
\end{align*} \]

Propose:

\[ \begin{align*}
    dx' &= adx + b dt' \\
    dt' &= cdx + e dt
\end{align*} \]

Restriction - \( a, b, c, e \) even in \( V \) - right or left boosts don't change (Isotropy of Space)

\[ \Delta t \text{ transforms as a scalar} \]
Yet another restriction

Reciprocity

\[ \text{origin, } x't' \text{ has velocity } -V \text{ in } x', t' \]

and another

Transformation by \( V \) \((x \rightarrow x')\)

Then by \(-V\) \((x' \rightarrow x)\)

This must be an identity \( V(V) = I \)

so with all restrictions

\[
\begin{pmatrix}
a & -b \\
-c & e
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & e
\end{pmatrix} = I \Rightarrow a^2 - be = 1
\]

\[ab - be = 0 \]
\[-ca + ec = 0 \]
\[-cb + e^2 = 1\]

We assume \( V = 1 \)

\( b = V e \) \( \text{see left} \)

\[
\begin{align*}
dx &= \frac{V_a}{\sqrt{2}} dt = \pm dt \\
dx' &= \frac{V_b}{\sqrt{2}} dt' = \pm dt' \\
dt' &= \pm dt
\end{align*}
\]

For this point

\[
\Rightarrow a + b = c + e \quad \text{light moves in same direction in both systems}
\]

For there to be simultaneously true

\[a = e + c = b \quad \text{remember } \frac{b}{c} = V \text{ so } \frac{c}{a} = V\]
\[
\begin{align*}
\begin{pmatrix}
(a & -b) \\
(c & e)
\end{pmatrix}
\begin{pmatrix}
(a & b) \\
(c & e)
\end{pmatrix}
&= I \\
a^2 - b^2 &= 1 \\
ab - be &= 0 \\
-ca + ce &= 0 \\
-bc + e^2 &= 1
\end{align*}
\]

\[
\begin{align*}
a &= e \\
c &= b \\
\frac{b}{e} &= \sqrt{1 - \frac{c^2}{a^2}} \\
e &= a = \frac{1}{\sqrt{1 - \nu^2}} \\
b &= \frac{\nu}{\sqrt{1 - \nu^2}} = c
\end{align*}
\]

\[
\begin{pmatrix}
\frac{1}{\sqrt{1 - \nu^2}} & -\frac{\nu}{\sqrt{1 - \nu^2}} \\
-\frac{\nu}{\sqrt{1 - \nu^2}} & \frac{1}{\sqrt{1 - \nu^2}}
\end{pmatrix}
\begin{pmatrix}
\gamma & \nu \gamma \\
-\nu \gamma & \gamma
\end{pmatrix}
\begin{pmatrix}
(x, t) \\
(-\nu \gamma, \gamma)
\end{pmatrix}
= \begin{pmatrix}
x' \\
-t'
\end{pmatrix}
\]

\[
\begin{align*}
x' &= \gamma (x + \nu t) \\
t' &= \gamma (t + \nu x)
\end{align*}
\]

\[
\gamma = \frac{1}{\sqrt{1 - \nu^2}}
\]

\[
\nu = \frac{1}{c^2}
\]

\[
\nu = \sqrt{1 - \frac{c^2}{v^2}}
\]

\[
\gamma = \frac{1}{\sqrt{1 - \nu^2}}
\]

\[
\nu = \frac{1}{c^2}
\]
Let's accclimate for a while at the elevated base camp.

1) Translation invariance demands that \( dx', dt \rightarrow dx', dt' \)

\[ \text{locally} \]

\( \text{at any point there exists a linear trans. that takes} \]
\( \Delta x = S \Delta x \]
\( \Delta x, \Delta t \rightarrow \Delta x', \Delta t' \]
\( \Delta x' = S \Delta x \)

\[ \text{globally} \]

translation has to be the same everywhere.

\[ \text{gives} \]
\[ \text{Transformation} \]
\[ \begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix} \]

\[ \text{Inverse} \]
\[ \begin{pmatrix} a - V a \\ -c \\ c - a \end{pmatrix} \begin{pmatrix} a & V a \\ c & e \end{pmatrix} = I \Rightarrow a^2 - V a c = 1 \]
\[ a (-V a) + e (V a) = 0 \]
\[ a = e \text{ unless } V a = 0 \]
\[ \Rightarrow V = 0 \text{ DULL!} \]

\[ \text{forall fixed point} \]
\( x = 0 \) \( \Delta t = a \Delta t' \)
\( \Delta x' = V \Delta t' \)

\[ \begin{align*}
\text{assume for a moment} & \quad \begin{pmatrix} a & V a \\ V a/k & a \end{pmatrix} \\
& \text{h as a constant}
\end{align*} \]

but not on velocity
E.M. \[
\begin{pmatrix}
  a & V_a \\
  V_a & s^2
\end{pmatrix} = \gamma \quad a^2 - V_a c = 1 \\
\gamma \quad a^2 \left(1 - \frac{V^2}{s^2}\right) = 1
\]

\[a = \frac{1}{\sqrt{1 - V^2/s^2}}\]

Just like Einstein except special velocity is \(s\).

\[V = V_a / \sqrt{1 - V^2/s^2}\]

(b) \(k'e < 0\) \(k' = 1\)

\[a^2 \left(1 + V^2\right) = 1 \quad a = \frac{1}{\sqrt{1 + V^2}}\]

\[b = \frac{V}{\sqrt{1 + V^2}}\]

We will see that ordinary orthogonal transformations represent a rotation between space + time.
We recall that we arrived at a most general transformation

\[
\begin{pmatrix}
x' \\
\frac{t'}{k}
\end{pmatrix} = \begin{pmatrix}
a & aV \\
\frac{aV}{k} & a
\end{pmatrix} \begin{pmatrix}
x \\
\frac{t}{k}
\end{pmatrix}
\]

really want \( k \) to be a constant

if \( k = c^2 \) we're saying that velocity of light is the same for everyone.

\[
\begin{pmatrix}
1 & \sqrt{1 + \frac{1}{c^2}} \\
\sqrt{1 + \frac{1}{c^2}} & 1
\end{pmatrix}
\]

so what if \( k = -1 \)

\[
\begin{pmatrix}
a & -aV \\
-aV & a
\end{pmatrix} \begin{pmatrix}
a & aV \\
-aV & a
\end{pmatrix} = I \implies a^2 (1 + \frac{1}{c^2}) = 1
\]

so

\[
a = \frac{1}{\sqrt{1 + \frac{1}{c^2}}} \quad aV = \frac{V}{\sqrt{1 + \frac{1}{c^2}}}
\]

set \( a = \cos \theta \) \( aV = \sin \theta \) \( a^2 (1 + \frac{1}{c^2}) = \sin^2 \theta + \cos^2 \theta = 1 \)

\[
V_x = \tan \theta \quad \theta = \frac{\pi}{2} \rightarrow V = \infty \quad \text{no speed limit}
\]

\[
x' = -ax - aV t \\
t' = +\frac{aV x - at}{k}
\]

If \((x,t)\) has \(2\) events \(q^+ \ x=0 \)

\[t_1 < t_2 \quad \text{In } x', t'_2 < t'_1 \]

in \( x', t' \) clocks run backward!!
Principle of causality can't be violated, no event can be influenced by a later event.

This absurd form corresponds to Euclidean space-time.

There is no distinction between space direction or time direction.

Rotations in 4-space all equivalent.

\[-c^2 + x^2 + y^2 + z^2 = \text{const}\]
\[+t^2 + x^2 + y^2 + z^2 = \text{const}\]

Going to \(+t^2\) while absurd, might make some calculations easier. Then you would transform back.

Example:

Euclidean Lattice Quantum Field Theory

\[e^{-iHt|\Psi_0\rangle}\]
\[\rightarrow e^{-i\chi}\]
\[\rightarrow e^{-c^2 + 140}\]

and you get an exponential fall off of time.
Meanwhile back in the real world

\[ k > 0 \]

with \[ a^2 (1-v^2) = 1 \] \[ \rightarrow a = \frac{1}{\sqrt{1-v^2}} \cos \gamma \]

\[ V_u = \frac{V}{\sqrt{1-v^2}} = \sinh \gamma \]

\( \gamma \) = rapidity

For \( dV \), \( dV = dy \)

\[ \frac{dV}{\cosh \gamma} = 1 \]

\[ \sinh d\gamma = dl \]

small angle approx

Compare and contrast Euclidean and hyperbolic

* Eucl. \( \Theta \rightarrow \infty \) but no difference from a value in range \((-\pi, \pi)\)

\( \leftrightarrow \) Periodicity of Boost transform.

\( \leftrightarrow \) Think of rotational symmetry since all axes (time+space) are equivalent.

* hyperbolic \( \gamma \in (-\infty, \infty) \) no 2 points are

\[ \Rightarrow \text{fixed point in } (x; t) \text{ never exceeds} \]

\[ V_0 = c \text{ in } \mathbb{R}^2 (x', t') \]

Why? \( \& \ V = \tanh \gamma \in (-1, 1) \)

\[ |\epsilon| = 1 \]
Space and time are now mixed (remember on Monday we didn't allow this), we can use Minkowski space under boost:

\[-t^2 + x^2 = -(t')^2 + (x')^2\]

In variants:

* Euclidean space 
  \[d\rho^2 = dx^2 + dy^2 + dz^2\]
  \(<space + time are orthogonal \rightarrow not mixed>\]

* Minkowski space
  \[-dt^2 + dx^2 + dy^2 + dz^2 = -(dt')^2 + (dx')^2 + (dy')^2 + (dz')^2\]

Recall \(E + p \neq \text{relation in Galilean transform}\):

\[
\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & aV \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \quad \Rightarrow a = 1
\]

\[
\begin{pmatrix} a & -aV \\ 0 & a \end{pmatrix} \begin{pmatrix} a & +aV \\ 0 & a \end{pmatrix} = I
\]

\[
\Rightarrow a^2 = 1
\]

we conclude that Einstein relativity goes to Galilean as \(c \rightarrow \infty\).

Recall \(\begin{pmatrix} aV \\ +aV/c^2 \end{pmatrix}\).