1. Jackson 4.8

There are 3 regions

(a) \( r > b, \, \epsilon_0 \)
(b) \( a < r < b, \, \epsilon \)
(c) \( r < a, \, \epsilon_0 \)

Let the axis of the cylinders be the \( z \)-axis. Take the applied field to point in the \( x \)-direction. That is, \( \vec{E}_0 = E_0 \hat{x} \) corresponding to a potential

\[
\phi_0 = -E_0 x = -E_0 r \cos \theta.
\]

The potential in each of the regions can be expanded as

\[
\begin{align*}
\phi_1 &= -E_0 r \cos \theta + A_0^{(1)} + B_0^{(1)} \ln r \\
&\quad + \sum_{n=1}^{\infty} \left\{ B_{n}^{(1)} r^{-n} \cos n\theta + D_{n}^{(1)} r^{-n} \sin n\theta \right\} \\
\phi_2 &= A_0^{(2)} + B_0^{(2)} \ln r + \\
&\quad + \sum_{n=1}^{\infty} \left\{ (A_{n}^{(2)} r^n + B_{n}^{(2)} r^{-n}) \cos n\theta + \\
&\quad \quad (C_{n}^{(2)} r^n + D_{n}^{(2)} r^{-n}) \sin n\theta \right\} \\
\phi_3 &= A_0^{(3)} + \sum_{n=1}^{\infty} \left\{ A_{n}^{(3)} r^n \cos n\theta + C_{n}^{(3)} r^n \sin n\theta \right\}
\end{align*}
\]

There is no \( \ln r \) term in \( \phi_3 \) because it would give a \( 1/r \) singularity in the electric field at the origin. Since the potential is defined only up to an overall constant, we can set \( A_0^{(1)} = 0 \).
The boundary conditions are:

\[ \phi_1(b, \theta) = \phi_2(b, \theta) \]
\[ \phi_2(a, \theta) = \phi_3(a, \theta) \]
\[ \epsilon_0 \frac{\partial \phi_1}{\partial r} \bigg|_{r=b} = \epsilon \frac{\partial \phi_2}{\partial r} \bigg|_{r=b} \]
\[ \epsilon \frac{\partial \phi_2}{\partial r} \bigg|_{r=a} = \epsilon_0 \frac{\partial \phi_3}{\partial r} \bigg|_{r=a} \]

Since \( \sin n\theta \) and \( \cos n\theta \) are orthogonal on \( 0 < \theta < 2\pi \), these equations must be satisfied separately by the coefficients of each trigonometric term (including \( 1 = \cos 0\theta \)). Except for the \( \cos \theta \) terms the boundary conditions give sets of 4 independent homogeneous equations in 4 unknowns. Only the trivial solutions with all coefficients equal to 0 are allowed.

For the \( \cos \theta \) terms we have inhomogeneous equations. Relabeling the coefficients

\[ B_1^{(1)} = B_1 \]
\[ A_1^{(2)} = A_2 \]
\[ B_1^{(2)} = B_2 \]
\[ A_1^{(3)} = A_3 \]

the boundary conditions can be written as

\[ B_1 - A_2 b^2 - B_2 = E_0 b^2 \]
\[ B_1 + \kappa A_2 b^2 - \kappa B_2 = -E_0 b^2 \]
\[ A_2 a^2 + B_2 - A_3 a^2 = 0 \]
\[ \kappa A_2 a^2 - \kappa B_2 - A_3 a^2 = 0 \]

where \( \kappa = \epsilon/\epsilon_0 \). The solution is

\[ B_1 = \frac{2(\kappa^2 - 1)(b^2 - a^2)}{\Delta} E_0 b^2 \]
\[ A_2 = \frac{-2(\kappa + 1)}{\Delta} E_0 b^2 \]
\[ B_2 = \frac{-2(\kappa - 1)a^2}{\Delta} E_0 b^2 \]
\[ A_3 = \frac{-4\kappa}{\Delta} E_0 b^2 \]

where \( \Delta = (\kappa + 1)^2 b^2 - (\kappa - 1)^2 a^2 \).

We have two checks on these results.

a) If \( \kappa = 1 \), there is no interface, and we obtain \( B_1 = B_2 = 0 \), and \( A_2 = A_3 = -E_0 \). Thus, \( \phi = -E_0 x \) everywhere.

b) If \( b = a \), \( B_1 = 0 \), and \( A_3 = -E_0 \).

In general we have

\[ \phi_1 = -E_0 x \left\{ 1 - \frac{\left[2(\kappa^2 - 1)(b^2 - a^2)b^2\right]}{\Delta} \left[ \frac{1}{x^2 + y^2} \right] \right\} \]
\[ \phi_2 = -E_0 x \left\{ \frac{2(\kappa + 1)b^2}{\Delta} + \frac{2(\kappa - 1)a^2 b^2}{\Delta} \left[ \frac{1}{x^2 + y^2} \right] \right\} \]
\[ \phi_3 = -E_0 x \left[ \frac{4\kappa b^2}{\Delta} \right] \]

Defining

\[ \vec{V} = \nabla \left[ \frac{x}{x^2 + y^2} \right] = \frac{(y^2 - x^2)\hat{x} - 2xy\hat{y}}{(x^2 + y^2)^2} , \]

the electric fields in the three regions are given by

\[ \vec{E}_1 = \vec{E}_0 - \frac{2(\kappa^2 - 1)(b^2 - a^2)b^2}{\Delta} E_0 \vec{V} \]
\[ \vec{E}_2 = \left[ \frac{2(\kappa + 1)b^2}{\Delta} \right] \vec{E}_0 + \left[ \frac{2(\kappa - 1)a^2 b^2}{\Delta} \right] E_0 \vec{V} \]
\[ \vec{E}_3 = \left[ \frac{4\kappa b^2}{\Delta} \right] \vec{E}_0 \]
4.8 b. For \( b = 2a \)

\[
\Delta = 4(k+1)^2 a^2 - (k-1)^2 a^2 = (3k^2 + 10k + 3) a^2
\]

\[
\vec{E}_1 = \frac{\vec{E}_0}{\Delta} - \left[ \frac{2(1-k)}{\Delta} \right] \vec{E}_0 \overrightarrow{V}
\]

\[
\vec{E}_2 = \left[ \frac{8(1+k)^2 a^2}{\Delta} \right] \vec{E}_0 + \left[ \frac{8(1-k)}{\Delta} \right] \vec{E}_0 \overrightarrow{V}
\]

\[
\vec{E}_3 = \left[ \frac{16k a^2}{\Delta} \right] \vec{E}_0.
\]

C. \( a \to 0 \)

\[
\Delta = (k+1)^2 b^2
\]

\[
\vec{E}_1 = \vec{E}_0 - \left[ \frac{2(k-1)}{(k+1)} \right] \vec{E}_0 \overrightarrow{V}
\]

\[
\vec{E}_2 = \left[ \frac{2}{(k+1)} \right] \vec{E}_0.
\]

\( b \to \infty \)

\[
\vec{E}_2 = \left[ \frac{2}{(k+1)} \right] \vec{E}_0 + \left[ \frac{2(1-k)}{(k+1)^2} \right] \vec{E}_0 \overrightarrow{V}
\]

\[
\vec{E}_3 = \left[ \frac{4k}{(k+1)^2} \right] \vec{E}_0.
\]
(a) & (b) Choose the z-axis to be perpendicular to the plane where the 2 hemispheres meet as shown above. Then we have axial symmetry and can expand the potential in Legendre polynomials in each half.

\[ \phi_j = A_0^{(j)} + B_0^{(j)}/r + \sum_{l=1}^{\infty} \left( A_l^{(j)} r^l + B_l^{(j)} r^{-l-1} \right) P_l(\cos \theta), \quad j = 1, 2. \]

At the conducting surfaces, \( r = a, \quad r = b \), the potentials are constant. Hence,

\[
\begin{align*}
A_0^{(j)} + B_0^{(j)}/a &= V_a \\
A_0^{(j)} + B_0^{(j)}/b &= V_b
\end{align*}
\]

while for \( l \neq 0 \)

\[
\begin{align*}
A_l^{(j)} a^l + B_l^{(j)} a^{-l-1} &= 0 \\
A_l^{(j)} b^l + B_l^{(j)} b^{-l-1} &= 0
\end{align*}
\]

For \( l \neq 0 \), all the coefficients are zero, and the potentials are

\[ \phi_j = A_0^{(j)} + B_0^{(j)}/r. \]
On the interface we have $\phi_1(r) = \phi_2(r)$ for $a < r < b$. Hence the coefficients are equal, and we have everywhere

$$\phi = A + B/r.$$ 

The electric field is

$$\vec{E} = \frac{-B}{r^2} \hat{r}.$$ 

To find the value of $B$ we apply the boundary condition for the normal component of the electric field. Let $\sigma_0$ and $\sigma$ denote the free surface charge densities at $r = a$ on the two hemispheres of the conductor.

The boundary condition is

$$E_n(a) = -B/a^2 = \sigma_0/\varepsilon_0 = \sigma/\varepsilon,$$

which implies

$$\varepsilon \sigma_0 = \varepsilon_0 \sigma.$$ 

Charge conservation gives

$$2\pi a^2 (\sigma + \sigma_0) = Q.$$ 

The solution to these equations is

$$\sigma_0 = \frac{\varepsilon_0 Q}{(\varepsilon + \varepsilon_0)2\pi a^2},$$

$$\sigma = \frac{\varepsilon Q}{(\varepsilon + \varepsilon_0)2\pi a^2},$$

and the field is

$$\vec{E} = \frac{-Q\hat{r}}{2\pi(\varepsilon + \varepsilon_0)r^2}.$$ 

(c) The total charge density at $r = a$ must be the same on both halves of the sphere since $\vec{E}$ is the same. Therefore,

$$\sigma + \sigma_p = \sigma_0$$

which gives

$$\sigma_p = \frac{-(\varepsilon - \varepsilon_0)Q}{(\varepsilon + \varepsilon_0)2\pi a^2}.$$
4.11. The Clausius-Mossotti relation reads,

\[ Y_{\text{mol}} = \frac{3}{N} \left( \frac{R-1}{R+2} \right), \quad R = \frac{E}{E_0}. \]

Since \( Y_{\text{mol}} \) is a constant for a given \( E \) field, by equation (2), \( \frac{R-1}{R+2} \) should be proportional to the density of the substance \( N \).

For air, one can get the following data from AIP Handbook.

<table>
<thead>
<tr>
<th>Pressure (atm)</th>
<th>( \frac{R-1}{R+2} )</th>
<th>( \frac{E}{E_0} )</th>
<th>( % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>56.72</td>
<td>7.21 x 10^{-3}</td>
<td>5089.43</td>
</tr>
<tr>
<td>100</td>
<td>91.61</td>
<td>1.79 x 10^{-2}</td>
<td>5106.75</td>
</tr>
</tbody>
</table>

For Pentane,

<table>
<thead>
<tr>
<th>Pressure (atm)</th>
<th>( \frac{R-1}{R+2} )</th>
<th>( \frac{E}{E_0} )</th>
<th>( % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.613</td>
<td>0.21466</td>
<td>0.82</td>
</tr>
<tr>
<td>10^3</td>
<td>0.701</td>
<td>0.24242</td>
<td>0.96</td>
</tr>
<tr>
<td>4 x 10^3</td>
<td>0.796</td>
<td>0.27184</td>
<td>1.12</td>
</tr>
<tr>
<td>8 x 10^3</td>
<td>0.865</td>
<td>0.29245</td>
<td>1.24</td>
</tr>
<tr>
<td>12 x 10^3</td>
<td>0.907</td>
<td>0.30716</td>
<td>1.33</td>
</tr>
</tbody>
</table>

Both for pentane and for air, the C-M relation hold approximately.

For air, the fractional variation is following.

\[ \frac{0.1161 - 36.72}{36.72} = 14.95\% \]

\[ R-1: \frac{0.0548 - 0.0218}{0.0218} = 151.4\% \]

So, \( P \) changes a little more apparent than the density \( S \).

For pentane, the fractional variation is

<table>
<thead>
<tr>
<th>pressure</th>
<th>( \alpha (R-1) % )</th>
<th>( \Delta S % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^3</td>
<td>17.07%</td>
<td>14.36%</td>
</tr>
<tr>
<td>4 x 10^3</td>
<td>26.59%</td>
<td>29.85%</td>
</tr>
<tr>
<td>8 x 10^3</td>
<td>51.22%</td>
<td>41.11%</td>
</tr>
<tr>
<td>12 x 10^3</td>
<td>62.20%</td>
<td>47.96%</td>
</tr>
</tbody>
</table>
Comparing (-M relation with cruder relation \( k - 1 \approx \rho \)), we find for the range considering, they both have a linear structure with \( \rho \) with similar precision.