PHYS 505

Homework No. 9

1.5.1

\[ dB = \frac{k_0 l}{4\pi} dl' + \frac{x - x'}{1 + x'^2} \]

\[ = \frac{k_0 l}{4\pi} dl' \hat{P}_o \frac{1}{1 + x'^2} \]

\[ B(x') = \frac{k_0 l}{4\pi} \int dl' \times \hat{P}_o \frac{1}{1 + x'^2} \]

Variation of Gauss' Theorem:

\[ \oint dl + \oint (\nabla \times A) = 0 \]

\[ = \oint \oint dl \times \hat{P}_o \frac{1}{1 + x'^2} \]

\[ = \oint dl + \oint \oint dl' \times \hat{P}_o \frac{1}{1 + x'^2} \]

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\[ = D \oint dl \times \hat{P}_o \frac{x - x'}{(x^2 + 1)} \]

\[ = D \oint dl \times \hat{P}_o \frac{x - x'}{(x^2 + 1)^{3/2}} \]

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For a loop of current \( i \) on a wire, below:

\[
B = \frac{\mu_0 i}{2} \left( \frac{a^2}{L^2 + x^2} \right)^{3/2}
\]

For total flux \( \Phi \) of coil:

\[
\Phi = \sum \frac{\mu_0 i}{2} \left( \frac{a^2}{L^2 + x^2} \right)^{3/2}
\]

\[
\lim_{N \to \infty} NL \to 0
\]

\[
\sum f(z) = N \int f(z) \, dz
\]

\[
B = \frac{\mu_0 N I}{2} \left( \int_{-Z}^{Z} \frac{a^2 \, dz}{L^2 + z^2} \right)^{3/2}
\]

Split integral into:

\[
\int_{-Z}^{Z} \frac{a^2 \, dz}{L^2 + z^2} = \int_{0}^{Z} \frac{a^2 \, dz}{L^2 + z^2} + \int_{-Z}^{0} \frac{a^2 \, dz}{L^2 + z^2}
\]

For \( Z > 0 \), let \( Z = a \cot \theta \)

\[
\int_{0}^{Z} \frac{a^2 \, dz}{L^2 + z^2} = \int_{\theta}^{\pi/2} \sin \theta \, d\theta = \cos \theta
\]

For \( Z < 0 \), let \( Z = a \cot \theta \)

\[
\int_{-Z}^{0} \frac{a^2 \, dz}{L^2 + z^2} = \int_{0}^{\theta} \sin \theta \, d\theta = \cos \theta
\]

Then

\[
B = \frac{\mu_0 N I}{2} \left[ \cos \theta_1 + \cos \theta_2 \right]
\]
Jackson 5.7 - Helmholtz Coils.

(a) \[ \oint d\alpha \times d\beta \] Let current loop lie in xy-plane with \( z = z_0 \) through center of circle. Use cylindrical coordinates \( \rho, \theta, z \).

(i) By axial symmetry \[ \vec{B}(\rho, \theta, z) = \vec{B}(\rho, \theta + \theta_0, z) \]

\[ \Rightarrow \text{no } \theta \text{ dependence} \]

(ii) Consider a circle parallel to xy-plane with center on z-axis at \( y = 2 \) with radius \( \rho \). No current passes through loop. Therefore,
\[ \nabla \cdot \vec{B} = 2 \pi \rho B_0(\rho, z) \Rightarrow B_0 = 0 \]

And \[ \vec{B} = \frac{\rho}{2} B_0(\rho, z) + \frac{1}{2} B_0(\rho, z) \]

(iii) It also follows by axial symmetry that on the z-axis,
\[ B_z = \frac{\rho}{2} B_0(0, z) \]
\( \text{(i.e. } B_0(0, z) = 0.} \)

\[ (d\vec{B})_z = \left( \frac{d\rho}{\mu_0} \right) \frac{d\rho}{\mu_0} \sin \alpha \]

\[ \Rightarrow \left( \frac{d\rho}{\mu_0} \right) \frac{d\rho}{\mu_0} \frac{q}{r^2} \]

\[ \Rightarrow B_z = \frac{\mu_0 I \rho q}{2 \mu_0 r^3}, r^2 \equiv [a^2 + 2^2]^{1/2} \]
(6) With the 2 coils arranged as described in the last, we have:

\[ B_2(0, z) = \frac{\mu_0 I a^2}{2} \left[ \left( \frac{x^2 + (z - b_0)^2}{(x^2 + (z - b_0)^2)^{3/2}} \right) + \left( \frac{x^2 + (z + b_0)^2}{(x^2 + (z + b_0)^2)^{3/2}} \right) \right] \]

We want to expand in powers of \( \frac{z}{d^2} \), \( d^2 = a^2 + b^2/4 \), so we write

\[ B_2(0, z) = \frac{\mu_0 I a^2}{2} \left\{ \left[ 1 - \frac{b_0}{d^2} + \frac{a^2 + b_0^2}{d^4} \frac{z^2}{3!} \right] - \frac{b_0}{d^3} + \frac{(a^2 + b_0^2)}{d^4} \frac{z^3}{3!} \right\} \]

When we expand these terms, odd powers of \( \frac{z}{d^2} \) cancel and even powers add up. To get the required accuracy of the formula in Faraday, we need a May/00 series up to \( z^6 \).

\[ 1 + x^3 - 3 \frac{x^2}{2} + \frac{1}{3!} \left( \frac{3 \cdot 5 - 3 \cdot 3}{2^3} \right) - \frac{1}{4!} \left( \frac{3 \cdot 5 - 3 \cdot 3}{2^4} \right) + O(\frac{1}{5}) \]

The algebra yields:

\[ B_2(0, z) = \left( \frac{\mu_0 I a^2}{d^3} \right) \left[ 1 + \frac{3(b^2 - a^2) z^2}{d^4} \right. \]

\[ + \frac{15(b^4 - 6b^2a^2 + 2a^4) z^6}{d^8} + O \left( \frac{z^8}{d^8} \right) \]

\]
Near the origin, we can expand \( B_z \) \& \( B_g \) up to 2nd order in the coordinates as:

\[
B_z (r, z) = 50 + 5r^2 z^2 + 2g + 6g^2 + 45^2
\]

\[
B_g (r, z) = 125 + 7g + 4g^2 + 75g^2
\]

To a 1st order given by part (b). The remaining constants, \( a, \ldots, e \), can be determined by applying the field equations \( \nabla \times B = 0 \), \( \nabla \cdot B = 0 \).

In this case, they become:

\[
\frac{\partial B_g}{\partial z} - \frac{\partial B_z}{\partial r} = 0 \quad \text{and} \quad \frac{1}{8} \frac{\partial^2 (g B_g)}{\partial z} + \frac{\partial B_z^2}{\partial z} = 0
\]

The 1st equation gives:

\[
e_5 = a + b r + 2c z
\]

Then, \( a = 0 \), \( b = 0 \), \( c = 2c \).

The 2nd equation gives:

\[
2d + 2e + 3f g = -25r^2 + 5g
\]

Hence, \( d = 0 \), \( 3fg = 0 \), \( e = -5z \), \( f = 2c \).

Put all this together, we have:

\[
B_z (r, z) = 50 + 5r^2 z^2 - 8g^2
\]

\[
B_g (r, z) = -5g z^2
\]

Note: If we use the same information \( B_z (r, z) = 50 + 5r^2 z^2 + 6g + 45^2 \), we can determine \( B (r, z) \) completely to 4th order by the same method.

Result: \( B_z (r, z) = 50 + 5r^2 (z^2 - 8z^2) + f_4 (24 - 3g^2 z + 7g^4) \)

\[
B_g (r, z) = -5g z^2 + f_4 (7/2 g^2 z^2 - 2g^2 z)
\]
(d) For an expansion at large \( z \), we can write \( \mu \) as
\[
B_2(0, z) = \frac{\cos \frac{\mu}{2}}{2!} z^2 \left[ (1 + \frac{a^2 + b^2}{2a^2} - \frac{b}{(2!)^2})^{-3/2} + (1 + \frac{a^2 + b^2}{2a^2} + \frac{b}{(2!)^2})^{-3/2} \right]
\]
Comparing to Eqn. \( \text{(B)} \), we see that the term above is obtained from \( \text{(B)} \) by the substitution \( c \to z \).

(i) For \( b = a \), the expression for the final \( B_2 \) becomes
\[
B_2 = \left( \frac{20 a^2}{a^2} \right) \left[ 1 - \frac{14 y}{125} (\frac{12}{a})^4 + \ldots \right]
\]

(ii) For an accuracy of \( 1 \times 10^{-4} \), we have
\[
\left( \frac{14 y}{125} \right) (\frac{12}{a})^4 = 10^{-4}
\]
so \( \frac{12}{a} < 10^{-1} \)

(iii) For an accuracy of \( 1 \times 10^{-2} \), we require \( \frac{12}{a} < 10^{-4} + 0.3 \)