PHYSICS 308 QUANTUM PHYSICS
MIDTERM EXAM 22 MARCH 2002

In the following, you should not be doing very long, complicated calculations. However, even if you are sure of an answer, don’t simply write it down, but also EXPLAIN YOUR REASONING.

1. Consider the Schrödinger equation $H\psi = E\psi$, with $H = -\hbar^2 \partial_x^2/2m + m\omega^2 x^2/2$.
   
   (a) Using the correspondence $-i\hbar \partial_x \sim p$, explain what classical mechanical system has this expression for its energy.
   
   (b) Check that $\psi_0(x) = \frac{4}{\sqrt{\pi \hbar}} \sqrt{m\omega} \ e^{-m\omega x^2/2\hbar}$ satisfies the equation for some $E = E_0$.
   
   (c) What is $E = E_1$ for the wave function $\psi_1 \propto x\psi_0$?

2. For some Hamiltonian $H = -\hbar^2 \partial_x^2/2m + V(x)$, two normalized bound state wave functions are $\psi_1$ and $\psi_2$, with $H\psi_1 = E_1\psi_1$ and $H\psi_2 = E_2\psi_2$. Suppose $\int dx |\psi_1|^2 x = \int dx |\psi_2|^2 x = 0$, and $\int dx \psi_1^* x \psi_1 = \ell$. Let the normalized wave function $\Psi = c_1 \psi_1 e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2 t/\hbar}$ be an equal superposition of $\psi_1$ and $\psi_2$. You may find it a helpful shortcut to define $\omega \equiv (E_2 - E_1)/\hbar$.
   
   (a) Calculate $\langle x \rangle(t)$ for $\Psi$.
   
   (b) Use the relationship between $p$ and $dx/dt$ to determine $\langle p \rangle(t)$ for $\Psi$.
   
   (c) Use Ehrenfest’s theorem to determine $\langle -\partial_x V \rangle(t)$ for $\Psi$.

SOLUTIONS

Grading is on the basis of 10 points per part, for a maximum possible grade of 60 points.

1. 
   
   (a) Substituting $-i\hbar \partial_x \rightarrow p$, we get $H = p^2/2m + m\omega^2 x^2/2$. The first term is the kinetic energy $mv^2/2$ for a particle of mass $m$, and the second term is the potential for a mass on a spring, because it leads to the restoring force $F = -kx = -\partial_x V$, with $V = kx^2/2$ and $k = m\omega^2$. This is the harmonic oscillator, with general solution $x = \ell \cos \omega(t + \phi)$, as can be checked by substituting into Newton’s equation $F = ma = md^2x/dt^2$. 
(b) The normalization constant is just a factor independent of \( x \), so ignore it.

\[
\partial_x^2 e^{-m\omega x^2/2 \hbar} = \partial_x \left( -m\omega x/\hbar \right) e^{-m\omega x^2/2 \hbar} = \\
\left[ (-m\omega x/\hbar)^2 - m\omega/\hbar \right] e^{-m\omega x^2/2 \hbar} .
\]

Therefore we have

\[
(-\hbar^2 \partial_x^2/2m)\psi_0 + (m\omega^2 x^2/2)\psi_0 = (-m\omega^2 x^2/2 + \hbar \omega / 2 + m\omega^2 x^2/2)\psi_0 ,
\]

or \( H\psi_0 = E_0\psi_0 = (\hbar \omega / 2)\psi_0 .
\]

(c) Now we need to compute \( H\psi_0 \), and see if this is equal to a constant multiple \( E_1 x\psi_0 \):

\[
\partial_x^2 x\psi_0 = \partial_x (\partial_x x\psi_0) = \partial_x \psi_0 + \partial_x x \partial_x \psi_0 = \\
2\partial_x \psi_0 + x \partial_x^2 \psi_0 .
\]

Using \( \partial_x \psi_0 = (-m\omega x/\hbar)\psi_0 \), we have \( Hx\psi_0 = xH\psi_0 + \hbar \omega x\psi_0 \), giving

\[
E_1 x\psi_0 = xE_0\psi_0 + \hbar \omega x\psi_0 = E_0 x\psi_0 + \hbar \omega x\psi_0 = 3\hbar (\omega/2)x\psi_0.
\]

2. (a) To make \( \Psi \) normalized we require \( |c_1|^2 + |c_2|^2 = 1 \), because \( \psi_1 \) and \( \psi_2 \) both are normalized, and orthogonal to each other [The orthogonality of wave functions with different energies was proved in class]. Equal superposition means \( |c_1|^2 = |c_1|^2 \), giving \( |c_1| = |c_2| = 1/\sqrt{2} \). Because we do not care about the overall phase of the wave function, and we may choose the starting time for convenience, assume \( |c_1| \) and \( |c_2| \) both are real and positive. Then we have \( c_1 = c_2 = 1/\sqrt{2} \). Now from the definitions we have

\[
\langle x \rangle = 2Re[c_1 c_2 e^{i(E_2 - E_1) t/\hbar} \int dx \psi_2^* x \psi_1] .
\]

[The contributions from \( |\Psi|^2 \) involving \( |\psi_i|^2 \) vanish by assumption, so only the cross terms involving \( \psi_2^* \psi_1 \) and \( \psi_2^* \psi_2 \) contribute.] From the definitions of \( \ell \) and \( \omega \), with \( c_1 c_2 = 1/2 \), we get \( \langle x \rangle = 2(1/2)\ell \cos \omega t = \ell \cos \omega t .\]

(b) We learned that the relationship between the operators \( p \) and \( dx/dt \) is \( p = mv = mdx/dt \), so \( \langle p \rangle(t) = md\langle x \rangle/dt = -m\omega \ell \sin \omega t .\]

(c) Ehrenfest’s theorem states \( d\langle p \rangle/dt = -\partial_x \langle V(x) \rangle \). For our case that gives \( -\partial_x \langle V(x) \rangle = -m\omega^2 \ell \cos \omega t = -m\omega^2 \langle x \rangle .\) This means that for a single pair of states combined in the way specified one can describe the expectation values of \( x(t) \) and its time derivatives as if the system were a harmonic oscillator with mass \( m \) and spring constant \( k = m\omega^2 \). This is just a fancy way of saying that \( \langle x \rangle \) has simple harmonic motion.