Exploring the uncertainty principle

1. Assume $\psi(x) = Ne^{-(x^2/2a^2)}$. Here $N$ is chosen to give $\langle \psi | \psi \rangle = 1$. Find $\langle \psi | x \psi \rangle$ and $\langle \psi | x^2 \psi \rangle$. Determine $\int dx e^{-(ipx/\hbar)} \psi(x)$ for $p = 0$.

2. Determine $\tilde{\psi}(p) = \tilde{N} \int dx e^{-(ipx/\hbar)} \psi(x)$ for general $p$. What is the value of $\tilde{N}$ if $\tilde{\psi}$ is normalized to unity?

3. Now obtain the reverse Fourier transform $f(x') \equiv \int dp e^{i(p x'/\hbar)} \tilde{\psi}(p)/\tilde{N}$. Use the ratio of $f(x)$ to $\psi(x)$ to deduce the Fourier integral theorem, $\int dk dx e^{ikx'} e^{-ikx} g(x) = 2\pi g(x')$ for any reasonably smooth function $g(x)$.

Orbital angular momentum

4. From the analogy with classical mechanics, we have the orbital angular momentum components $L_z = xp_y - yp_x, L_x = yp_z - zp_y, L_y = zp_x - xp_z$. Show $[L_x, L_y] = i\hbar L_z$. This suggests (correctly!) $[L_y, L_z] = i\hbar L_x$ and $[L_z, L_x] = i\hbar L_y$.

5. Define $L_{\pm} \equiv L_x \pm L_y$. Show $[L_z, L_{\pm}] = \pm\hbar L_{\pm}$. 