The construction of supergravity theories and their symmetries.

- Review the methods to construct a supergravity theory.

- Show that $D = 11$ supergravity is a non-linear realization. \texttt{hep-th/05270}

- Argue that $D = 11$, $\text{IIA}$ and $\text{IIB}$ supergravity theories have a large Kac-Moody symmetry ($E_{11}$). \texttt{hep-th/0104081 0107181 0107209}
Methods for Constructing Supergravity Theory

- Noether method using an action
- Noether method using the algebra
- On-shell superspace
- Dimensional reduction

- Gauging a space-time group
  - Chen-sedding, W.T.
  - MacDowell, M.

- Off-shell superspace
  - Wess, Zumino
  - Siegel
The Noether method using an action

Construction of off-shell supergravity

Start from the linearized theory

$h_{\mu\nu}, \Psi_{\mu\alpha}; M, N, B$

The action

$$A^{(0)} = \int d^4x \left[ h \Box h + \Psi \Box \Psi + h^2 + N^2 + B^2 \right]$$

is invariant under

$$8 h_{\mu\nu} = \varepsilon \, \bar{Y}_\mu \Psi_\nu,$$

and

$$8 \Psi_{\mu\alpha} = \partial_\mu \bar{Y}_\alpha.$$

Let $\varepsilon \rightarrow \varepsilon(\lambda^\mu)$, then

$$8 A^{(1)} = \int d^4x \left[ \partial_\mu \varepsilon \bar{Y}_\mu \Psi_{\lambda\nu} \right] j^\nu_{\lambda\alpha} = \left( \partial_\mu \Psi + \ldots \right) \bar{Y}_\mu j^\nu_{\lambda\alpha}.$$

An action invariant to order $\lambda^0$ is

$$A^{(1)} = A^{(0)} - \frac{1}{2} \int d^4x \, \Psi_{\mu\alpha} j^\mu_{\lambda\alpha}.$$

provided

$$\eta_{\lambda\alpha} \approx \frac{1}{\lambda^2} \varepsilon(\lambda)_{\lambda\alpha}.$$

Now add to action and transformations to gain invariance order by order in $\lambda$. 
\[ A = \int d^{4}x \left\{ \frac{e}{2 \kappa^2} R - \frac{1}{2} \psi_{\mu} \mathcal{R}^{\mu} - \frac{e}{3} M^{2} + N^{2} - b \psi_{\mu} \right\} \]

invariant under

\[ \delta \psi_{\mu} = \frac{2}{\kappa^2} D_{\nu} \xi + i \gamma_{5} \left( \xi_{\mu} - \frac{1}{3} \chi_{\mu} \right) \xi \]

\[ - \frac{1}{3} \chi_{\mu} \left( M + i \gamma_{5} N \right) \xi \]

e tc
Constructions of \( \Pi B \) supergravity

Field content

\[ e^{\alpha}, A, A_{\mu}, A_{\mu, \nu}, \psi_{\mu \alpha}, \lambda \]

The linearized equations of motion are invariant under the rigid supergravity transformation

\[ \delta \lambda = \gamma^{\mu} e_{\mu} \lambda \epsilon^* + \ldots \]

\[ \delta A = \epsilon^* \lambda + \ldots \]

Let \( \epsilon \rightarrow \epsilon(x) \) then

\[ [\delta \epsilon_1, \delta \epsilon_2] \lambda_{\alpha} = \epsilon_1^* \gamma^{\mu} \epsilon_2 e_{\mu} \lambda_{\alpha} \]

\[ + \gamma^{\mu} (e_{\mu} \lambda_1^* \epsilon_2 - (1 \leftrightarrow 2) \rho A = 0 \]

Closure to order \( \kappa^0 \) is given by

\[ S \lambda = \gamma^{\mu} (e_{\mu} \lambda - \frac{3}{2} \Theta \lambda) \epsilon^* \]

Provided \( 6 \gamma_{\mu \alpha} = \frac{3}{2} e_{\mu} \epsilon + \ldots \)

Proceeding order by order in \( \kappa \) one finds the field transformation and equations of motion of \( \Pi B \) supergravity.
On-shell Superspace.

The IIB supergravity content is
\[ e^a, A, A_{\mu}, A_{\nu}, \ldots, m, \Phi_{\alpha}, \Lambda_{\lambda} \]
\[ \phi^\mu, \phi^\nu, \phi_{\mu \nu}, \phi_{\mu \nu}, \phi_{\mu \nu \lambda}, \phi_{\mu \nu \lambda \kappa} \]
\[ \text{dimension} \quad 2 \quad 1 \quad 1 \quad 3/2 \quad 1/2 \]
\[ \text{on gauge change} \quad 0 \quad 4 \quad 2 \quad 0 \quad 1 \quad 3 \]

Superspace \( X^a, \Theta^a, \Theta^\alpha \) contains
the supervielbein \( E_m^a \) and super connection
\( \Omega_{m}^{ab} \). The invariant objects are
\[ T_{AB}^{c} = E_A^{m} \Omega_{m}^{E_B^{\nu}} E_B^{c} - \Omega_{A^{c}}^{B^{c}} \]
and \( R_{AB}^{cd} \).

On grounds of \( U(1) \) charge and dimension
\[ T_{\alpha}^{c} = 0, \quad T_{\alpha}^{\alpha} = 0 = T_{\alpha}^{\bar{\alpha}} \]
The Bianchi identities then solve for the torsions and curvatures in terms of a few superfields and give the equations of motion.

\[ T_{a b} \bar{\chi} = \delta^{b}_{c} \Lambda_{c} \]

The Bianchi

\[ D_{a} T_{b} \bar{\chi} + \ldots = 0 \]

\[ \sigma_{a} D_{a} \Lambda + \ldots = 0 \]

We identify

\[ A_{c} \theta = 0 = \lambda_{c}(x^a), \ldots \]

\[ \sigma_{a} D_{a} \tilde{\Lambda} = \frac{i}{16.2} \sigma_{a b c d e} \Lambda (\Lambda \sigma_{a b c d e}) + \frac{5}{8.192} \sigma_{a b c d e} \Lambda G_{a b c d e} \]

\[ G_{a b c d e} = \sigma_{a b c d e} - \Lambda \sigma_{a b c d e} \Lambda \]

where

\[ G_{m_{1} m_{2} \ldots m_{5}} = 5 \bar{\Theta}_{m_{1}} \bar{\Theta}_{m_{2}} \ldots \bar{\Theta}_{m_{5}} + 60 \iota (\bar{\Theta}_{m_{1}} \bar{\Theta}_{m_{2}} \bar{\Theta}_{m_{3}} \bar{\Theta}_{m_{4}} \bar{\Theta}_{m_{5}} - \cdots) \]

\[ - 20 \bar{\Psi}_{c a} \sigma_{b c d} \Psi_{c} \]

Howe Wait.
Scalars in Supergravity Theory

The two scalars of $N=4$, $D=4$ supergravity belong to the coset $SU(1,1)/U(1)$.

The non-linear realization of $G$ with subgroup $H$, given $g(x^\mu) \in G$, the theory is invariant under

$$g(x^\mu) \rightarrow g_0 g(x^\mu) h(x^\mu)$$

$g_0 \in G$, $h \in H$ rigid local.

The Cartan forms transform as

$$v \equiv g^{-1} dg \rightarrow h^{-1} v h$$

Let the generators of $G$ be $T^a$ and $h_i \in H$ then we can use the local $H$ invariance to choose

$$g(x^\mu) = \exp \phi^a(x^\mu) T^a$$

If $G$ involves $P^\mu$ then

$$g = e^{x^\mu P_\mu} e^{\phi^a(x^\mu) T^a}.$$
The scalars in supergravity theories belong to cosets.

- IIB has 2 scalars in $SU(1,1)/U(1)$

- $D=11$ supergravity reduced on a torus:

  - $D=10$, $\mathbb{Z}/2\mathbb{Z}$
  - $D=9$, $G_2$ (2)
  - $D=6$, $E_6 \cong SO(5,5)$
  - $D=5$, $E_6$ (USp(6) x USp(6))
  - $D=4$, $E_7$ (USp(8))
  - $D=3$, $E_8$ (SU(8))
  - (Cremmer, Julia, de Wit, Nicolai)

- The gauge fields can also be included in a non-linear realization (Cremmer, Julia, de Wit, Nicolai)
Kac–Moody Algebras

Given a generalized Cartan matrix $A_{ab}$ such that

(i) $A_{aa} = 2$
(ii) $A_{ab}$ for $a \neq b$ are negative integers or zero
(iii) $A_{ab} = 0 \iff A_{ba} = 0$

and a set of Chevalley generators $E_a, F_a, H_a$ which obey the same relations

$[H_a, H_b] = 0; \quad [H_a, E_b] = A_{ab} E_b$

$[H_a, F_b] = -A_{ab} F_b; \quad [E_a, F_b] = \delta_{ab} H_a$

and

$[E_a, [E_a, \ldots, [E_a, E_b] \ldots]] = 0$

1. Aba sector

plus relations for $F_a$.

Then the Kac–Moody algebra is given by the multiple commutator

$[E_{a_1}, [E_{a_2}, \ldots, [E_{a_{n-1}}, E_{a_n}] \ldots]]$

plus multiple commutator of $F_a$'s.

It is uniquely determined by $A_{ab}$.
If $A_{ab}$ is a positive definite matrix we get a finite dimensional semi-simple Lie algebra.

Example $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

The Chevalley generators are $E_1, E_2, H_1, H_2, F_1, F_2$.

The Cartan relations are

$[H_1, E_2] = -E_2$ etc

and

$[E_1, [E_1, E_2]] = 0$ etc.

The resulting algebra is

$E_1, E_2, [E_1, E_2]$ \{ positive roots \}

$(\kappa_1) (\kappa_1) (\kappa_1 + \kappa_2)$

$H_1, H_2$ Cartan subalgebra

$F_1, F_2, [F_1, F_2]$ \{ negative roots \}

$(-\kappa_1)(-\kappa_2)(-\kappa_1 - \kappa_2)$

i.e. $\mathfrak{su}(3)$. 
field \( e^a, A_{a_1 a_2 a_3}, A_{a_1 ... a_6} \)

generator \( K_{a b}, R_{a_1 a_2 a_3}, R_{a_1 ... a_6} \).

These obey an algebra \( G_{11} \) which contains \( \text{GL}(11) (K_{a b}) \), \( P^a \) and

\[
\left[ K_{a b}, R^{c_1 ... c_3} \right] = \delta^c_b R^{a c_2 c_3} + ... \\
\left[ R^{c_1 ... c_3}, R^{c_4 ... c_6} \right] = 2 R^{c_1 ... c_6}
\]

We consider the non-linear realization of \( G_{11} \) with subgroup \( \text{SO}(1, 10) \) \( \subset \)

\[
g = e^{x_a \epsilon_a} e^{h_{a b} (x) K_{a b}} \\
ex \left\{ \frac{1}{3} A_{c_1 ... c_3} R^{c_1 ... c_3} + \frac{1}{6} A_{c_1 ... c_6} R^{c_1 ... c_6} \right\}
\]

- carry out simultaneous non-linear realization with conformal group \( \text{SO}(2, 11) \).

The bosonic equations of motion of \( D = 11 \) supergravity i.e.

\[
F_{a_1 a_2 ... a_9} = \frac{1}{2!} \epsilon_{a_1 ... a_9} \quad \Omega_{a_1 a_2 ... a_9}
\]
The Borel Subalgebra of $E_7$.

Restricting the indices to $i = 5, ..., 11$ in $G_{11}$ gives the subalgebra

$$D, \; K^i_5, \; R^i_{\cdot \cdot \cdot -11}, \; S^i_6 = \frac{1}{6!} E^i_{\cdot \cdot \cdot -16} R^i_{\cdot \cdot \cdot -16}$$

This is the Borel subalgebra of $E_7$.

The adjoint of $E_7$ breaks to $SL(7)$ as

$$133 = 48 (K^i_5) + 1 (D) \quad G_{11}(7)$$

$$+ \begin{array}{c}
35 (R^i_{\cdot \cdot \cdot -3}) + 7 (S^i_3) \\
+ 35 (R^i_{\cdot \cdot \cdot -3}) + 7 (S^6_3)
\end{array} \quad \text{the roots}$$

$D=11$ supergravity is invariant under the Borel subalgebra of $E_7$. We increased the local subgroup so as to include $SU(8)$ in the restriction we would have a full $E_7$ symmetry.
Is $D=11$ supergravity invariant under a Kac-Moody symmetry?

$G_{11}$ is not a Kac-Moody algebra, but we can

(i) use an alternative description of $D=11$ supergravity

(ii) enlarge the local symmetry

$g \rightarrow g \, gh \, g$ does not affect field content

such that the resulting closure with the conformal algebra is a Kac-Moody algebra.

The local subalgebra should be the one invariant under the Cartan involution.

\[ g = \prod e^{H_\alpha} \prod e^{\phi_\alpha} e^{E_\alpha} \]
Identification of Kac-Moody Algebra \( G \).

We split the generators in \( G_{II} \) into

\[
G_{II}^+ = \sum_{a < b} \kappa^a \kappa^b, \quad R^{01,03}, R^{01,03'}
\]

\[
G_{II}^0 = \sum_{a} \kappa^a \kappa^{a+1}, a = 1, \ldots, 10, \quad D = \sum \frac{E}{a} \kappa^a
\]

The remaining generators are \( \kappa^a \kappa^b \), \( a > b \) and are in the local subgroup

We expect that \( G_{II}^+ \subset G^+ \).

Now \( G_{II}^+ \) is generated by

\[
E_a = \kappa^a \kappa^{a+1}, a = 1, \ldots, 10 \text{ and } E_{11} = R^{91,101}
\]

and identify these as the simple roots of \( G \).

We expect that \( G_{II}^0 \subset H \) and so a rank 11 algebra.

After choice of an appropriate basis for \( H \) to we can determine \( G \) from

\[
[H_a, E_b] = \delta_{ab} E_b.
\]
Now $E_a$ for $a = 1, \ldots, 10$ are the simple roots of $A_{10}$ and $E_a$ for $a = 5, \ldots, 11$ are the simple roots of $E_7$.

Given these embeddings the unique Kac-Moody algebra has a Cartan subalgebra

$$H_a, a = 1, \ldots, 10, \quad H_{11} = \kappa_q + \kappa_{10} + \kappa_{11} - \frac{1}{3} D$$

and is $E_{11}$.
\( G_{11} \) does not contain \( E_8 \). Under \( se(8) \) the adjoint of \( E_8 \) decomposes as

\[
248 \rightarrow 63 + 1 + 56 + 28 + 8 + 56 + 28 + 8
\]

\[
\chi, \ \Phi \quad R^{k_1,k_2} \quad \varepsilon_{k_1,k_2} \quad R^{11} \quad \text{negative roots}
\]

\[
\text{GL(8)} \quad \text{positive roots}
\]

\( i, j = 4, \ldots, 11 \)

where is the \( 8 \)?

We can modify the \( G_{11} \) algebra by adding a new generator \( R^{a_1 \ldots a_8, b} \)

- gives Borel subalgebra of \( E_8 \)

\[
R^{c_1 \ldots c_8} \quad R^{c_1 \ldots c_6} \quad ; \quad K^{a \ b} \quad R^{a_1 \ldots a_8, b}
\]

\[
A_{c_1 \ldots c_6} \quad A_{c_1 \ldots c_6} \quad h_{a \ b} \quad h_{a \ldots a_8, b}
\]

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dual formulation of 3-form

dual formulation of \( g_{\mu \nu \rho} \).
Conclusions

- We have argued that $D = 11$ supergravity is invariant under $E_{11}$.

- A similar calculation implies that $\Pi A$ and $\Pi B$ are also $E_{11}$ invariant.

\[ \rightarrow \quad \Pi A \quad \Pi B \quad D = 11 \]

($D = 11$ to $\Pi A$ requires enlarged $G_{11}$ algebra)

- The low energy effective action of the $D = 26$ closed bosonic string $(h_{ab}, \phi, B_{a12})$ has a corresponding Kac-Moody algebra of rank 27.

\[ \cdots \quad 26 \quad \cdots \quad 25 \]

- Requires a new formulation of gravity which should explain Geroch symmetry.

- Conjecture that M-theory is invariant under $E_{11}$. 