Renormalization: introduction by $\phi^3$

The $\phi^3$ Lagrangian in $D$ dimensions is taken as

$$\mathcal{L}_D = \frac{1}{2} \left( (\partial_\mu \phi)^2 - m^2 \phi^2 \right) - \frac{g\mu^\varepsilon}{3!} \phi^3,$$

where for applications to $D = 4$ we choose $\varepsilon = 2 - D/2$, which gives $g$ units of mass for $D = 4$. For $\mathcal{L}_{6-2\varepsilon}$ we take $\varepsilon = 3 - D/2$, which makes $g$ dimensionless for $D = 6$.

The mass counterterm, scheme and scale

In the self energy, Eq. (32), we see an additive, momentum-independent ultraviolet divergence, represented by a simple pole in $\varepsilon = (1/2)(4 - D)$. Now the Fourier transform of a constant is a delta function, so in coordinate space the divergence is local. As a result, it can be cancelled by a new vertex. Such a vertex can cancel the $p^2$-independent pole, but will not affect the momentum dependence otherwise. This is the procedure we’re after – to modify the theory in such a way as to make its ill-defined predictions for constant terms finite, while preserving its sensible predictions for momentum dependence.

We implement this procedure by introducing a new vertex in the theory, modifying the Lagrange density. Specifically, we add a new term (and also change the notation for the mass and coupling slightly, attaching a subscript ‘$R$’, for “renormalized”),

$$\mathcal{L}_{\text{ren}} = \frac{1}{2} \left( (\partial_\mu \phi)^2 - m^2_{\text{R}} \phi^2 \right) - \frac{gR\mu^\varepsilon}{3!} \phi^3 - \frac{1}{2} \delta m^2 \phi^2 \mathcal{L}_{\text{class}} + \mathcal{L}_{\text{ctr}}$$

where $\mathcal{L}_{\text{ctr}}$ is a mass counterterm,

$$\mathcal{L}_{\text{ctr}} = -\frac{1}{2} \delta m^2 \phi^2,$$

with

$$\delta m^2 = -\frac{g^2}{2(4\pi)^2} \left( -\frac{1}{\varepsilon} + c_m \right) m^2_{\text{R}}.$$

At this point, the constant $c_m$ and the renormalization mass $\mu$ are still arbitrary.
• In (44), the constant $c_m$ expresses the fact that the theory makes no prediction on the $\Sigma$ at any value of $p^2$, only on differences. A choice of $c_m$, then simply corresponds to the baseline value we are going to choose for the self-energy. The choice of $c_m$ is called a renormalization scheme.

• $\delta m^2$ is called the mass shift.

• $m_R$ is the renormalized mass. It is not necessarily the same as the physical mass, at which $\Gamma_2 = 0$.

• $g_R$ is the renormalized coupling, which in this case is just a way of denoting the coupling with integer units of mass, independent of $D$.

The renormalized two-point function, is now found from the full one-loop perturbation expansion, as the sum of the original self-energy diagram, Eq. (33) plus the counterterm, which gives

$$
\Sigma_{\text{ren}} = \frac{g^2}{2(4\pi)^2} \left\{ c_m - \gamma_E + 2 + \ln \left( \frac{4\pi\mu^2}{m_R^2} \right) - \sqrt{1 - \frac{4m^2}{p^2}} \ln \left[ \frac{\sqrt{1 - \frac{4m^2}{p^2}} - 1}{\sqrt{1 - \frac{4m^2}{p^2}} + 1} \right] \right\} 
+ \mathcal{O}(\varepsilon).
$$

(45)

Schemes

Perhaps the most common choice is the $\overline{\text{MS}}$ or ‘modified minimal subtraction’ scheme, specified by,

$$
c_{\overline{\text{MS}}} = -\ln 4\pi + \gamma_E,
$$

which cancels terms, which result from the Gamma function and the one-loop momentum integrals. If $c_m = 0$, the scheme is called ‘MS’ or simply ‘minimal subtraction’. These ‘minimal’ schemes specify the value of $\Sigma$ only indirectly. Such terms are, in fact, ubiquitous in one-loop calculations using dimensional regularization, so this is a very natural choice to make.

Other conventional choices are called ‘momentum subtraction’ schemes, where we specify the value of $\Sigma(p^2, m_R^2)$ directly, for example, we can specify that $\Sigma(p_0^2, m_R^2) = 0$, so that $\Gamma_2(p_0^2 - m_R^2) = p_0^2 - m_R^2$. If we choose in addition $p_0^2 = m_P^2$ with $m_P$ the physical mass, then the renormalized mass is fixed to be the physical mass and we have what is known as an ‘on-shell’ momentum subtraction scheme.
The tadpole diagram in $\phi^3$

There are only three diagrams in $\phi^3$ with superficial degree of divergence $\geq 0$, the self energy, and the tadpole diagrams at order $g$ and $g^3$. For the latter we have (with a conventional $-i$ for the 1PI function),

$$\gamma_1 = \frac{g}{2(4\pi)^2} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(\varepsilon - 1)m^2$$

which is again $p^2$-independent. The integral, proportional to $\int d^Dk \, (k^2 - m^2 + i\varepsilon)^{-1}$, is finite only for $\varepsilon > 1$, that is $D < 2$, but as usual can be extended to all $D$ by using the expression we get in dimensional continuation. Note that the delta function continues to give only a simple pole for $D = 4$, and that the continued integral is actually finite for $D = 3$.

All order-$g$ tadpole diagrams are automatically cancelled by a counterterm $-\tau\phi(x)$, with $\tau$ chosen as above. The $g^3$ tadpole can be cancelled in the same way. Note that $\tau$ is real. Of course, this is a specific choice of renormalization scheme, which we make for simplicity.

Summary for $\phi^3_D$

$$\mathcal{L}_{\text{ren}} = \mathcal{L}_{\text{KG}} + \frac{g_3}{3!} \phi^3 - \delta m^2(c_m) \phi^2 - \tau \phi.$$ 

With the choices for $\delta m^2$ and $\tau$ given above, perturbation theory generated from this Lagrange density is completely finite, order by order in $g$. Every divergent subdiagram will be matched with a counterterm, and the result will be free of poles in $\varepsilon$, although of course it will still depend on our choices of $\mu$ and the scheme (that is, $c_m$).

The key to renormalization: bare and renormalized Lagrangians

What are we to do with this finite perturbation theory? The method for $\phi^3_4$ illustrates the general procedure.

- We begin by choosing a scheme ($c_m$) and a renormalization mass ($\mu$).
• From the explicit expression for $\Sigma$, Eq. (33), we find the position of the physical mass for this $c_m$ and $\mu$, by solving

$$\Gamma_2(p^2 = m_P^2) = m_P^2 - m_R^2 - \Sigma(m_P^2, m_R, g_R) = 0, \quad (46)$$

which ensures that to this order in perturbation theory, $G_2(p^2) = i/\Gamma_2(p^2)$ has a pole at $p^2 = m_P^2$.

• We then “measure” (more realistically choose) a physical value of $m_P$, something like 1 GeV or 1 MeV.

• We then solve Eq. (46) for $m_R(\mu)$ as a function of $g_R$ (which in this superrenormalizable case we can also just specify).

• With this value of $m_R(\mu)$ and $g_R$ we can now go ahead and compute every other physical quantity and correlation function in the theory to the same order in $g_R$. In this example, one calculation and one “measurement” is all we need to get a completely finite perturbative expansion.

We can further interpret the procedure by the following reorganization of the Lagrange density,

$$\mathcal{L}_{\text{ren}}(m_R, g_R \mu^\epsilon, c_m) = \frac{1}{2} \left( (\partial_{\mu}\phi)^2 - m_R^2 \phi^2 \right) + \frac{g_R \mu^\epsilon}{3!} \phi^3 - \delta m^2(c_m) \phi^2$$

$$= \frac{1}{2} \left( (\partial_{\mu}\phi)^2 - (m_R^2 + \delta m^2(c_m)) \phi^2 \right) + \frac{g_R \mu^\epsilon}{3!} \phi^3$$

$$= \frac{1}{2} \left( (\partial_{\mu}\phi)^2 - m_0^2 \phi^2 \right) + \frac{g_0}{3!} \phi^3$$

$$= \mathcal{L}_{\text{class}}(m_0, g_0), \quad (47)$$

where in the third line we define the bare mass, $m_0^2 = m_R^2 + \delta m^2$ and the bare coupling, $g_0 = g_R \mu^\epsilon$. The final line is once again the classical Lagrangian, now in terms of $m_0$ and $g_0$, which we refer to as the bare Lagrangian (density). We summarize this result as:

**The renormalized Lagrangian = The bare Lagrangian.**

We can summarize all of our calculations above as: From the renormalized Lagrangian, we can generate a finite perturbation series; its equality to the
bare Lagrangian ensures that this series depends upon only two parameters, and therefore only requires two inputs. In this case, the inputs are a “measurement” of the physical mass and a specification of the coupling.

**Renormalization for $\phi^3_R$**

The classical Lagrange density for $\phi^3_R$ is the same as for $\phi^3_4$, except that we let $g_R$ be dimensionless, so that it now appears as $g_R\mu^\varepsilon$, with $\varepsilon = 3 - D/2$.

Renormalization will follow the same pattern, but now, because $[g] = 0$, the superficial degree of divergence of each $\Gamma_E$ depends only on $E$, not on the order, $V$ (see Eq. (40)). Only for $E = 2$ and $E = 3$ will we need counterterms; the tadpoles, $E = 1$, are treated as in four dimensions.

The self energy ($E = 2$) is now

$$\Sigma(p^2,m^2) = -\frac{g^2}{2(4\pi)^3} \left\{ \left( -\frac{p^2}{6} + m^2 + \frac{1}{\varepsilon} \right) \right. $$

$$+ \left( -\frac{p^2}{6} + m^2_R \right) \left( 1 + \ln \frac{4\pi\mu^2 e^{-\gamma}}{m^2_R} \right) $$

$$-m^2_R \int_0^1 dx F(x) \ln F(x) + O(\varepsilon) \}$$

(48)

where

$$F(x) = 1 - x(1-x)p^2/m^2_R.$$ 

The poles appear only times $p^2$ and $m^2_R$.

For the vertex correction (triangle diagram, with $E = 3$), the finite, $\varepsilon^0$ are more complex because they depend on two independent external momenta, but here we need only the pole terms, given by

$$\Gamma_3 = (g_R\mu^\varepsilon) \frac{g^2}{2(4\pi)^3} \left( \frac{1}{\varepsilon} + \ln \left( 4\pi e^{-\gamma} \right) \right) + O(\varepsilon^0),$$

(49)

which is independent of the external momenta.

The UV poles (divergences for $D = 6$) in Eqs. (48) and (49) are cancelled by three new counterterms, which we add to the classical Lagrange density in the following rather unintuitive notation as

$$L_{\text{ren}}(m_R, g_R\mu^\varepsilon) = \frac{1}{2} \left( (\partial_\mu \phi_R)^2 - m^2_R \phi_R^2 \right) - \frac{g_R\mu^\varepsilon}{3!} \phi_R^3 $$

$$+ \frac{1}{2} \left( Z_\phi - 1 \right) (\partial_\mu \phi_R)^2 - \delta m^2 \phi_R^2 - \frac{\delta g^2}{3!} \mu^\varepsilon \phi_R^3 $$

(50)
The $\delta m^2$ and $Z_\phi - 1$ counterterms automatically cancel the $p^2/\varepsilon$ and $m^2/\varepsilon$ poles of the self energy, respectively. The $\delta g$ counterterm cancels the pole in the vertex function. One-loop calculations in the theory are then all finite, although still functions of $c_m$ and $\mu$. Notice that now the field as well as the mass and coupling has subscript ‘R’, denoting renormalization. Unlike $\phi_4^3$, two- and higher-loop renormalization is now necessary, beginning with all the two-loop diagrams that we get from the classical Lagrange density plus counterterms computed to one loop. Although the calculations are more elaborate, the steps in the method are the same. Only the renormalization constants $Z_{\phi,g,m}$ are necessary, and are themselves expansions in the (square of) the coupling,

$$Z_i = 1 + \sum_{j \geq 1} z_i^{(j)}(\varepsilon) g_R^{2j}$$

(51)

where the coefficients $z_i^{(j)}$ are constructed to cancel poles that appear in $j$-loop diagrams (i.e., at order $g_R^{2j}$).

**Reprise of the method**

We summarize the renormalization of $\phi_6^3$ following what we did for $\phi_4^3$.

- Pick a scheme.
- Calculate two (at least) physical quantities, $\sigma_i^{(calc)}(m_R, g_R)$. One of these could be the position of the physical mass, $m_p$, as we did for $\phi_4^3$, but this is not the only choice (both here and for $\phi_4^3$).
- Measure the $\sigma_i^{(exp)}$ (or postulate values).
- Set $\sigma_i^{(exp)} = \sigma_i^{(calc)}$.
- Solve for $m_R$ and $g_R$ as functions of $\mu$ in this scheme.
- Can now calculate all other quantities in the theory to the accuracy in which $m_R$ and $g_R$ have been determined (that is, the original order of $\sigma_i^{(calc)}$).
Multiplicative renormalization

We get further insight by relating bare to renormalized quantities, including now the field and coupling, as well as the mass.

\[ \begin{align*}
\phi_R Z^{1/2} &= \phi_0 \\
m_R^2 Z_m &= m_0^2 \\
g_R \mu \phi Z_g &= g_0 \\
\delta m^2 &= m_R^2 (Z_m Z_{\phi} - 1) \\
\delta g &= g_R \mu \phi (Z_g Z_{\phi}^{3/2} - 1)
\end{align*} \]

\[ \mathcal{L}_{\text{ren}}(m_R, g_R \mu) = \frac{1}{2} \left( (\partial_{\mu} \phi_R)^2 - m_{\phi R}^2 \right) - \frac{g_R \mu \phi^3}{3!} + \frac{1}{2} \left( Z_{\phi} - 1 \right) (\partial_{\mu} \phi_R)^2 - \left( Z_m Z_{\phi} - 1 \right) m_{\phi R}^2 \phi_R^2 - \left( Z_g Z_{\phi}^{3/2} - 1 \right) \frac{g_R \mu \phi^3}{3!} \]

\[ \equiv \frac{1}{2} \left( (\partial_{\mu} \phi_0)^2 - m_0^2 \phi_0^2 \right) - \frac{g_0 \mu \phi^3}{3!} \]

And once again, we summarize this as

\[ \mathcal{L}_{\text{ren}}(g_R, M_R, c_i, \mu) = \mathcal{L}_{\text{class}}(g_0, m_0) \]

where the left hand side generates a finite perturbation expansion in \( g_R \) and \( m_R \) (which depends on \( c_m, c_{\phi}, c_g \) and \( \mu \)), while from the right hand side we learn that this expansion still depends on only two independent parameters.

Summary of one-loop counterterms

Defining \( a = \frac{g_R}{2(2\pi)^{3/2}} \), we have from the one-loop results quoted above in Eqs. (48) and (49),

\[ \begin{align*}
\delta m^2 &= -\frac{a}{2} \left( \frac{1}{\varepsilon} + c_{\delta m} \right) \\
Z_{\phi} - 1 &= -\frac{a}{12} \left( \frac{1}{\varepsilon} + c_{\phi} \right) \\
\delta g &= g \mu \phi \left[ -\frac{a}{2} \left( \frac{1}{\varepsilon} + c_{\delta g} \right) \right]
\end{align*} \]
from which we derive

\[ Z_m - 1 = -\frac{5a}{12} \left( \frac{1}{\varepsilon} + c_m \right) \]
\[ Z_g - 1 = -\frac{3a}{8} \left( \frac{1}{\varepsilon} + c_g \right) \]

where we can readily determine \( c_{\delta m} \) and \( c_{\delta g} \) from \( c_{m,g,\phi} \) that appear in the \( Z_i \).